RANK 3 PERMUTATION CHARACTERS AND MAXIMAL SUBGROUPS

by

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Abstract

Let G be a transitive permutation group acting on a finite set $\mathfrak{E}$. Let P be a stabilizer in G of a point in $\mathfrak{E}$. We say G is primitive rank 3 on $\mathfrak{E}$ if P is maximal in G and P has exactly three orbits on $\mathfrak{E}$. For any subgroup H of G, we denote by 1_H^G the permutation character (or permutation module over $\mathbb{C}$) of G on the cosets G/H. Let H and K be subgroups of G. We say $1_H^G \leqslant 1_K^G$ if $1_K^G - 1_H^G$ is either 0 or a character of G. Also a finite group G is called nearly simple primitive rank 3 if there exists a quasi-simple group L such that $L/Z(L) \trianglelefteq G/Z(L) \leqslant Aut(L/Z(L))$ and G acts as a primitive rank 3 permutation group on the set of cosets of a subgroup of L. In this thesis we classify all maximal subgroups M of a class of nearly simple primitive rank 3 groups G acting on $\mathfrak{E}$ such that $1_P^G \leqslant 1_M^G$, where P is a stabilizer of a point in $\mathfrak{E}$. This result has an application to the study of minimal genus of algebraic curves which admit group actions.

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Contents

Chapter 1

Introduction

Let G be a transitive permutation group acting on a finite set $\mathfrak{E}$. Let P be a stabilizer of a point in G. We say that G is *primitive* on $\mathfrak{E}$ if and only if P is maximal in G. We define the *rank* of G on $\mathfrak{E}$ to be the number of P-orbits on $\mathfrak{E}$. For any subgroup H of G, we denote by 1_H^G the permutation character of G on the cosets G/H. We also use the same notation 1_H^G for the permutation module. Let H, K be subgroups of G. Consider the permutation characters 1_H^G and 1_K^G, we say $1_H^G \leqslant 1_K^G$ if $1_K^G - 1_H^G$ is zero or a character of G. In terms of permutation modules, $1_H^G \leqslant 1_K^G$ if 1_H^G is isomorphic to a submodule of 1_K^G. A finite group L is said to be *quasi-simple* if L is perfect and $L/Z(L)$ is simple. A finite group G is called *nearly simple of type* L if $L \trianglelefteq G$ and $L/Z(L) \leqslant G/Z(L) \leqslant Aut(L/Z(L))$ for some quasi-simple group L. Moreover a finite group G is called *almost simple of type* L if $L \trianglelefteq G \leqslant Aut(L)$ for some finite simple group L. It follows from definitions that if G is nearly simple of type L then $G/Z(L)$ is almost simple of type $L/Z(L)$. Assume that $m \geqslant 3$ is an integer. Let L be one of the following quasi-simple groups $\Omega_{2m+1}(3), \Omega_{2m}^{\varepsilon}(3), \Omega_{2m}^{\varepsilon}(2)$ or $SU_m(2)$. Let G be a nearly simple group of type L such that G acts on the L-orbit $\mathfrak{E}(V)$ of non-singular points in the natural module V for L. Then G is a primitive rank 3 group on $\mathfrak{E}(V)$ with socle L. (Theorem A.1 in Appendix A). In this situation, we say that G is a *nearly simple primitive rank* 3 *group of type* L. Now fix groups L, G as above and also fix a stabilizer P in G of a non-singular point $\langle x \rangle$ in V. In this work, we classify all maximal subgroups M of G such that $1_P^G \leqslant 1_M^G$.

Theorem 1.1 *Let L be one of the following groups $\Omega_{2m+1}(3)$, or $\Omega_{2m}^{\varepsilon}(3)$, with $m \geqslant 3$, and G be a nearly simple primitive rank 3 group of type L. Let P be the stabilizer of a non-singular point in V. Let M be any maximal subgroup of G. Then $1_P^G \leqslant 1_M^G$ unless the pairs (L, M) appear in Tables 1.1-1.3.*

Remark of Theorem 1.1. For all pairs (L, M) in Table 1.1 and 1.2, 1_P^G is not contained in 1_M^G. The pairs in Table 1.3 are the cases that we have not determined whether or not there is containment. A similar result for nearly simple primitive rank 3 group of type $\Omega_{2m}^{\varepsilon}(2)$, $SU_m(2)$ and almost simple groups of sporadic type can be found in Appendix B.

The main motivation for this work comes from algebraic curves which admit group actions. From Riemann's Existence Theorem, we know that for every finite group there are infinitely many Riemann surfaces with automorphism group G. We would like to identify G-curves of smallest possible genus.

Now Theorem 1.1 and Corollary 8.2 in [12] yield the following corollary.

Corollary 1.2 *Assume the assumption and notations of Theorem 1.1. Let X be a compact Riemann surface with $Aut(X) \cong G$, and inertia groups $\langle g_1 \rangle, \cdots, \langle g_r \rangle$ over X/G. Then $g(X/P) \leqslant g(X/M)$ for any maximal subgroup M of G which does not appear in Tables 1.1-1.3.*

A triple $\mathfrak{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{P}, \mathcal{B}, \mathcal{I}$ are sets with $\mathcal{P} \cap \mathcal{B} = \emptyset$ and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ is called an *incidence structure.* The elements of $\mathcal{P}, \mathcal{B}$ and $\mathcal{I}$ are called *points, blocks* and *flags*, respectively. If $(p, b) \in \mathcal{I}$ then we say p and b are incident and write $p\mathcal{I}b$. If there exists a finite group G of incidence preserving permutations of $\mathcal{P}$ and $\mathcal{B}$ which is transitive on pairs $(p, b) \in \mathcal{P} \times \mathcal{B}$ with $p\mathcal{I}b$, then $\mathfrak{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is said to have a *flag transitive automorphism groups.* Such an incidence structure can arise from subgroups of G as follows. Let P, M

be subgroups of G and let $\mathcal{P}, \mathcal{B}$ be the sets of right cosets of P, M, respectively. We define an incidence structure $\mathcal{I}$ by $Px \ \mathcal{I} \ My$ if and only if $Px \cap My \neq \emptyset$. We now fix orderings of $\mathcal{P}$ and $\mathcal{B}$. The group G defines permutation representations π_1 and π_2 on $\mathcal{P}$ and $\mathcal{B}$, respectively by right multiplications. An *incidence matrix* of $\mathfrak{S}$ denoted by $\mathfrak{M} = \mathfrak{M}(\mathfrak{S})$ is a matrix whose rows and columns are indexed by $\mathcal{P}$ and $\mathcal{B}$, respectively with entries 1 on elements of $\mathcal{I}$ and 0 elsewhere. As G preserves the incidence structure, we have for any $g \in G$, $\pi_1(g)\mathfrak{M}\pi_2(g)^t = \mathfrak{M}$. Thus if $\mathfrak{M}$ has maximal rank then $\pi_1 \leqslant \pi_2$ or equivalently $1_P^G \leqslant 1_M^G$ as $\pi_1 = 1_P^G$ and $\pi_2 = 1_M^G$ (Lemma 2.3 in [33]). Now let X be a finite set of size n, and let e, f be integers such that $1 \leqslant e \leqslant f \leqslant n-e$ and denote by X_e, and X_f the collections of subsets of X of size e and f, respectively. Let $\mathfrak{S}_{ef}(X)$ be the incidence structure $(X_e, X_f, \mathcal{I})$ where $p\mathcal{I}b$ if $p \subseteq b$ for any $(p, b) \in X_e \times X_f$. Let V be an $\mathbf{F}_q$ vector space of dimension n with q a prime power. The collection of e-subspaces of V is denoted by V_e. Similarly one can form the incidence structure $\mathfrak{S}_{ef}V = (V_e, V_f, \mathcal{I})$. These two structures share many properties. For example, Kantor [27] showed that whenever $1 \leqslant e \leqslant f \leqslant n-e$, the incidence matrices of $\mathfrak{S}_{ef}(X)$ and $\mathfrak{S}_{ef}(V)$ have maximal rank. This can be interpreted as the relation between a group with BN-pairs and its Weyl groups. In [33], Lehrer generalized this observation to all classical groups and he managed to show that if G is any finite classical group then $1_{P_e}^G \leqslant 1_{P_f}^G$, whenever $1 \leqslant e < f \leqslant n-e$ except when $G = O_{2n}^+(q)$ and $e = 1, f = n-1$, where P_e, P_f denote the parabolic subgroups which are the stabilizers of a totally singular or isotropic subspaces of dimension e, f, respectively and n is the dimension of any maximal totally singular (isotropic) subspaces of G. (Theorem 5.1[33]). These results in fact follow from the statements about its Weyl groups and the passage between these two structures are provided by generic algebras introduced by Tits (Theorem 3.2[33]).

The permutation character containment can be determined by the rank of suitable incidence matrix as above. Now if G is an almost simple group which is doubly transitive on $\mathfrak{E}$ with point stabilizer P, then either $1_P^G \leqslant 1_M^G$ or $G = PM$ for any maximal subgroup M of G. As the maximal factorization of almost simple groups was classified completely by M. Liebeck, C. Praeger and J. Saxl in [34], we can tell exactly for which M we have the containment $1_P^G \leqslant 1_M^G$. In case of rank 3, M. Aschbacher, R. Guralnick and K. Magaard ([2]) have a criterion in terms of the Higman rank 3 parameters. In that paper, they consider the case when G is a nearly simple classical group acting on the set of singular points on its natural module. Also a partial result of this case has been dealt with by D. Frohardt, the second and the third authors above in [12]. In the case when G is a nearly simple primitive rank 3 group of sporadic type, the containment of the permutation characters of G is completely determined since all the permutation characters of maximal subgroups of G are stored in [13], except $HS.2, Fi_{22}.2$ and $Fi_{24}'.2$.

We now describe our strategy. Let L be a finite simple classical group of degree $d \geqslant 2$, defined over a finite field $\mathbf{F} = \mathbf{F}_q, q$ a prime power, and let V be the natural module for L. Assume that G is an almost simple group with simple socle L. We have a powerful theorem on the subgroup structure of G by M. Aschbacher. The theorem says that if M is a subgroup of G then M is either belongs to a collection $\mathcal{C}(G)$ of geometric subgroup of G or $M \in \mathcal{S}(G)$, that is, M is an almost simple group and the full covering group of the socle of M acts absolutely irreducible on the natural module V for G and cannot be realized over any proper subfield. Thus if M is a maximal subgroup of G then either $M \in \mathcal{C}(G)$ or $M \in \mathcal{S}(G)$. The subgroup structure and the maximality among members of $\mathcal{C}(G)$ has been determined by P. Kleidman and M. Liebeck in [29] when the degree is at least 13. For this case, using the geometrical properties of the groups, we can solve the problem completely. When M is not a geometric subgroup, that is, $M \in \mathcal{S}(G)$, the problem is much more complicated as we still do not know which members of $\mathcal{S}(G)$ are maximal. Now assume that $M \in \mathcal{S}(G)$. Denote by S the socle of M. So S is a non-abelian finite simple group. According to the Classification of Finite Simple Groups, S is an alternating group of degree at least 5, a finite group of Lie type or one of the 26 sporadic groups. By way of contradiction, we assume that $1_P^G \not\leq 1_M^G$. From this assumption, we will get an upper bound for the dimension of V in terms of the size of the automorphism group of S. From the definition of members in $\mathcal{S}(G)$, the full covering group $\widehat{S}$ of S acts absolutely irreducible on V. Now using the information on the lower bound for the dimension of the absolutely irreducible representations of finite simple groups, we will get a finite list of cases that we can handle either by constructing the representations or by computer program GAP [13].

As in the almost simple doubly transitive case, we can get a list of maximal subgroups M such that $1_P^G \not\leq 1_M^G$. In Table 1.1, we list all the cases when $M \in \mathcal{C}(G)$, Table 1.2 contains all cases when $M \in \mathcal{S}(C)$, and in the last table, we list the cases that we have not determined whether or not there is containment. Notice that we only have a finite number of exceptions in Table 1.3. Also there is a finite number of cases in Table 1.2.

Chapter 2 is the preliminaries, in this chapter, we will fix some notations and give some definitions as well as some properties of finite simple groups and also some information on representations of these groups. The proof of Theorem 1.1 is carried out in Chapter 3. We will deal with each type of quasi-simple groups separately.

For the notations in the tables of Theorem 1.1, the columns 'orbits' give the number of orbits of M on $\mathfrak{E}(V)$ and this is also the number of double cosets of group G on P and M. The first columns are the type of the nearly simple group G. The last columns 'Ref' give the references for the result. For example 3.18 means that the case is dealt with in Proposition 3.18. Other notations will be explained in Chapter 3.

Table 1.1: $M \in \mathcal{C}$

L	type of M	conditions	Remarks	orbits	Ref
$\Omega_{2m+1}(3)$	$O_1(3) \perp O^{\varepsilon}_{2m}(3)$				3.10
	P_α	$1 \leqslant \alpha \leqslant m$		$\leqslant 2$	3.13
	$O_1(3) \wr S_n$	$(n, \xi, r) = (5, \pm, t)$	$n = 2m+1$	2	3.17
		$(7, +, t)$		2	
	$O_\alpha(3^3)$		$2m+1 = 3\alpha$	3	3.20
$\Omega^{\varepsilon}_{2m}(3)$	$O_1(3) \perp O_{2m-1}(3)$				3.37
	P_α	$1 \leqslant \alpha \leqslant m$		$\leqslant 2$	3.38
	$O_1(3) \wr S_n$	$(\varepsilon, r) = (-, t)$,	$n = 6$	2	3.39
		$(\varepsilon, r, \xi) = (-, t, \square)$	$n = 10$	3	
	$O_m(3) \wr S_2$	$(\varepsilon, r) = (-, t)$,	m odd	3	3.41
	$GL_m(3).2$			1	3.42
	$O^+_{2a}(3^3)$	$r = s$	$m = 3a$	3	3.45
	$O^+_m(3^2)$		m is even	2	
	$O^-_{2a}(3^3)$	$r = t$	$m = 3a$	3	3.46
	$O^-_m(3^2)$	$\overline{G} = \overline{I}$	m is even	2	
	$O_m(3^2)$	$\overline{G} = \overline{I}$ if $\varepsilon = +$	m odd	2	3.47
	$GU_m(3)$			1	3.48
	$Sp_2(3) \otimes Sp_m(3)$	$\varepsilon = +$	m even	1	3.53

Table 1.2: $M \in \mathcal{S}$

L	socle of M	modules	Remarks	orbits	Ref
$\Omega_7(3)$	A_9	$\lambda = (8, 1)$		$\leqslant 2$	3.26
$\Omega_7(3)$	$PSp_6(2)$			2	3.27
$\Omega_{13}(3)$	$PSp_6(3)$	λ_2		$\leqslant 2$	3.30
$\Omega_7(3)$	$G_2(3)$	λ_1, λ_2		1	3.32
$\Omega_{25}(3)$	$F_4(3)$	λ_4		$\leqslant 2$	
$\Omega^-_{n-1-\varepsilon_3(n)}(3)$	$A_n, n = 5, 6, 7$	$\lambda = (n-1, 1)$		$\leqslant 2$	3.57
$\Omega^+_{10}(3)$	A_{12}		$\xi = \square$	2	
$\Omega^-_{16}(3)$	A_{18}		$\xi = \square, r = t$	4	
$\Omega^-_6(3)$	$L_3(4)$			1	3.60
$\Omega^+_8(3)$	$\Omega^+_8(2)$			$\leqslant 2$	
$\Omega^-_{10}(3)$	$\Omega_5(3)$	$2\lambda_1$	$r = t$	5	3.65
$\Omega^+_8(3)$	$\Omega_7(3)$	Spin module		1	
$\Omega^+_{16}(3)$	$\Omega_9(3)$	Spin module		1	
$\Omega^+_{24}(3)$	$2.Co_1$	Leech lattice		2	3.69

Table 1.3: Exceptions

L	socle of M	modules	Remarks	Ref
$\Omega_{41}(3)$	$S_8(3)$	λ_1		3.30
$\Omega_{77}(3)$	$E_6(3)$	adjoint module		3.32
$\Omega_{133}(3)$	$E_7(3)$	adjoint module		
$\Omega_{52}^{+}(3)$	$F_4(2)$			3.60
$\Omega_{2m}^{\varepsilon}(3)$	$PSp_{2\ell}(3)$	$\lambda_{\ell-1}$	$\ell = 5, 6, 7$	3.66
$\Omega_{36}^{-}(3)$	$PSp_6(3)$	$(1+3^i)\lambda_\ell$	$1 \leqslant i \leqslant 2$	
$\Omega_{52}^{\varepsilon}(3)$	$F_4(3)$	adjoint module		3.68

Chapter 2

Preliminaries

2.1 Finite classical groups

We adopt the constructions and notations of [29]. We denote by V a vector space of dimension n over the field $\mathbf{F}$, where $\mathbf{F}$ is either a finite field or an algebraically closed field of characteristic p. The group of all non-singular $\mathbf{F}$-linear transformations of V is denoted by $GL(V,\mathbf{F})$, the *general linear group* of V over $\mathbf{F}$. The *special linear group* of V over $\mathbf{F}$, $SL(V,\mathbf{F})$, is the group of all elements of $GL(V,\mathbf{F})$ with determinant 1. Fix a basis $\beta = \{v_1, v_2, \cdots, v_n\}$ of V. For $g \in GL(V,\mathbf{F})$, g_β denotes the $n \times n$ matrix which satisfies $v_i g_\beta = \Sigma_{j=1}^{n} (g_\beta)_{ij} v_j$. For $\lambda_i \in \mathbf{F}^*$, $i = 1 \ldots n$, we denote by $\mathrm{diag}_\beta(\lambda_1, \ldots, \lambda_n)$ the *diagonal* linear transformation which satisfies $v_i \mathrm{diag}_\beta(\lambda_1, \ldots, \lambda_n) = \lambda_i v_i$. The element $\mathrm{diag}_\beta(\lambda, \ldots, \lambda)$, with $\lambda \in \mathbf{F}^*$, is called a *scalar linear transformation,* or a *scalar.* The center of $GL(V,\mathbf{F})$ is the group of all non-zero scalars, which is isomorphic to $\mathbf{F}^*$. We will write $PGL(V,\mathbf{F})$ for the *projective general linear group* $GL(V,\mathbf{F})/\mathbf{F}^*$. For any subgroup of $GL(V,\mathbf{F})$, we write PX for the corresponding projective group $X/X \cap \mathbf{F}^*$. We also use the *bar* convention to denote reduction modulo scalars, for example, $\overline{GL(V,\mathbf{F})} = PGL(V,\mathbf{F})$. A map g from V to itself is called an $\mathbf{F}$-*semilinear transformation* of V if there is a field automorphism $\sigma(g) \in Aut(\mathbf{F})$ such that for all $v, w \in V$ and $\lambda \in \mathbf{F}$,

$$(v + w)g = vg + wg \text{ and } (\lambda v)g = \lambda^{\sigma(g)}(vg) \tag{2.1}$$

An $\mathbf{F}$-semilinear transformation g is *non-singular* if $\{v \in V | vg = 0\} = \{0\}$. We now define $\Gamma L(V, \mathbf{F})$ to be the set of all non-singular $\mathbf{F}$-semilinear transformations of V. It is a group, called the *general semilinear group* of V over $\mathbf{F}$. The map σ from $\Gamma L(V, \mathbf{F})$ to $Aut(\mathbf{F})$ is a surjective homomorphism with kernel $GL(V, \mathbf{F})$. As $\mathbf{F}^* \trianglelefteq \Gamma L(V, \mathbf{F})$, we can factor out the scalars to get the *projective general semilinear group* $P\Gamma L(V, \mathbf{F})$.
With the basis β as above, each element $g \in \Gamma L(V, \mathbf{F})$ is determined by its action on β together with $\sigma(g)$. If $\alpha \in Aut(\mathbf{F})$ then we define $\phi_\beta(\alpha)$ as the unique element of $\Gamma L(V, \mathbf{F})$ which lies in $\sigma^{-1}(\alpha)$ and fixes each v_i. Then

$$(\sum_{i=1}^{n} \lambda_i v_i)\phi_\beta(\alpha) = \sum_{i=1}^{n} \lambda_i^\alpha v_i \tag{2.2}$$

A *left linear form* on V is a map $\mathbf{f} : V \times V \to \mathbf{F}$ such that for each $v \in V$, the map $V \to \mathbf{F}$ given by $x \mapsto \mathbf{f}(x, v)$ is a linear map. There is an analogous definition for *right linear form.* A *bilinear form* is a map which is both a left linear and a right linear form. For any map $Q : V \to \mathbf{F}$, define $\mathbf{f}_Q : V \times V \to \mathbf{F}$ by $\mathbf{f}_Q(v, w) = Q(v + w) - Q(v) - Q(w)$. The map Q is called a *quadratic form* if $Q(\lambda v) = \lambda^2 Q(v)$ for all $v \in V$ and $\lambda \in \mathbf{F}$, and $\mathbf{f}_Q$ is a bilinear form. When Q is a quadratic form, $\mathbf{f}_Q$ is called the *associated bilinear form.* If $\mathbf{f}$ is a map form $V \times V$ to $\mathbf{F}$ and $\beta = \{v_1, v_2, \ldots, v_n\}$ is any basis for V, then we define $\mathbf{f}_\beta$ as the matrix satisfying $(\mathbf{f}_\beta)_{ij} = \mathbf{f}(v_i, v_j)$. For $v \in V$, we call $\mathbf{f}(v, v)$ the $\mathbf{f} - norm$ of v. We write (v, w) instead of $\mathbf{f}(v, w)$. Let κ be either a left linear form or a quadratic form. Then κ is a map from V^k to $\mathbf{F}$, where $k = 1$ or 2. We have $\kappa(\lambda v_1, \ldots, v_k) = \lambda^{3-k}\kappa(v_1, \ldots, v_k)$.

Assume that $(V, \mathbf{F}, \kappa)$ and $(V', \mathbf{F}, \kappa')$ are two spaces of dimension n over $\mathbf{F}$, where κ and κ' are either both left linear of both quadratic forms. An *isometry* is an invertible element

$g \in Hom_{\mathbf{F}}(V, V')$ which satisfies $\kappa'(\mathbf{v}g) = \kappa(\mathbf{v})$ for all $\mathbf{v} \in V^k$. If such an isometry exists, then $(V, \mathbf{F}, \kappa)$ and $(V', \mathbf{F}, \kappa')$ are said to be *isometric* and we write $(V, \mathbf{F}, \kappa) \cong (V', \mathbf{F}, \kappa')$. An invertible element $g \in Hom_{\mathbf{F}}(V, V')$ is a *similarity* if there exists $\lambda \in \mathbf{F}^*$ such that $\kappa'(\mathbf{v}g) = \lambda\kappa(\mathbf{v})$ for all $\mathbf{v} \in V^k$. If such a similarity exists, then $(V, \mathbf{F}, \kappa)$ and $(V', \mathbf{F}, \kappa')$ are said to be *similar.* In the case $(V, \mathbf{F}, \kappa) = (\mathbf{V}', \mathbf{F}, \kappa')$, the isometries are called κ-*isometries* and the set of all κ-isometries forms a subgroup of $GL(V, \mathbf{F})$. This subgroup is called κ-*isometry* group and is denoted by $I(V, \mathbf{F}, \kappa)$. An element of $I(V, \mathbf{F}, \kappa) \cap SL(V, \mathbf{F})$ is a *special κ-isometry.* The *special κ-isometry* group is denoted by $S(V, \mathbf{F}, \kappa)$. The set of all κ-similarities forms the similarity group, denoted by $\Lambda(V, \mathbf{F}, \kappa)$. (This is $\Delta(V, \mathbf{F}, \kappa)$ in [29].) Finally an element $g \in \mathit{\Gamma} L(V, \mathbf{F})$ is called a *κ-semisimilarity* if there exist $\lambda \in \mathbf{F}^*$ and $\alpha \in Aut(\mathbf{F})$ such that

$$\kappa(\mathbf{v}g) = \lambda\kappa(\mathbf{v})^{\alpha} \text{ for all } \mathbf{v} \in V^k. \tag{2.3}$$

The set of all κ-semisimilarities forms a group, denoted by $\Xi(V, \mathbf{F}, \kappa)$. (This group is denoted by $\mathit{\Gamma}(V, \mathbf{F}, \kappa)$ in [29].) Suppose that κ is surjective and $g \in \Xi(V, \mathbf{F}, \kappa)$ satisfying (2.3). Then λ in (2.3) is uniquely determined by g and the map $\tau : \Xi(V, \mathbf{F}, \kappa) \longrightarrow \mathbf{F}^*, \tau(\mathbf{g}) = \lambda$ is well-defined. The restriction of τ to $\Lambda(V, \mathbf{F}, \kappa)$ is a homomorphism to $\mathbf{F}^*$ with kernel $I(V, \mathbf{F}, \kappa)$. The field automorphism α in (2.3) is exactly $\sigma(g)$ given in (2.1). The restriction of σ to $\Xi(V, \mathbf{F}, \kappa)$ is a homomorphism to $Aut(\mathbf{F})$ with kernel $\Lambda(V, \mathbf{F}, \kappa)$.

Assume that $\mathbf{f}$ is a map from $V \times V$ to $\mathbf{F}$. Then $\mathbf{F}$ is called *non-degenerate* if for each $v \in V \setminus 0$, the maps from $V \to \mathbf{F}$ given by $x \mapsto (x, v)$ and $x \mapsto (v, x)$ are non-zero. A quadratic form Q is called *non-degenerate* if its associated bilinear form $\mathbf{f}_Q$ is non-degenerate. The map $\mathbf{f}$ is *symmetric* if $\mathbf{f}(v, w) = \mathbf{f}(w, v)$ for all $v, w \in V$. The map $\mathbf{f}$ is *skew - symmetric* if $\mathbf{f}(v, w) = -\mathbf{f}(w, v)$ for all $v, w \in V$. The map $\mathbf{f}$ is *symplectic* if $\mathbf{f}$ is skew-symmetric, bilinear, and $\mathbf{f}(v, v) = 0$ for all $v \in V$. Finally, $\mathbf{f}$ is said to be *unitary* if $\mathbf{F}$ has an involutory field automorphism α, $\mathbf{f}$ is left linear and

$\mathbf{f}(v,w) = \mathbf{f}(w,v)^{\alpha}$ for all $v, w \in V$.

Let $\mathbf{F}$ be a finite field of characteristic p, and let κ be a left linear form $\mathbf{f}$ or a quadratic form Q appearing in one of the four cases below:

case $\mathbf{L}$: κ is identically 0;

case $\mathbf{S}$: $\kappa = \mathbf{f}$, a non-degenerate symplectic form;

case $\mathbf{O}$: $\kappa = Q$, a non-degenerate quadratic form;

case $\mathbf{U}$: $\kappa = \mathbf{f}$, a non-degenerate unitary form.

We define

$$A = \begin{cases} \Xi\langle\iota\rangle & \text{in case } \mathbf{L} \text{ with } n \geqslant 3 \\ \Xi & \text{otherwise} \end{cases}$$

$$\Omega = \begin{cases} \text{certain subgroup of index 2 in S} & \text{in case } \mathbf{O} \\ S & \text{otherwise} \end{cases}$$

where ι is an inverse-transpose automorphism of $GL(V, \mathbf{F})$. Hence we get a chain of groups

$$\Omega \leqslant S \leqslant I \leqslant \Lambda \leqslant \Xi \leqslant A. \tag{2.4}$$

This chain is A-invariant and $\mathbf{F}^* \trianglelefteq \mathbf{A}$, so we can define projective groups for any subgroups of A. We use the notation $\ddot{}$ to denote reduction modulo $\Omega\mathbf{F}^*$ in A. We get a chain

$$1 = \ddot{\Omega} \leqslant \ddot{S} \leqslant \ddot{I} \leqslant \ddot{\Lambda} \leqslant \ddot{\Xi} \leqslant \ddot{A}. \tag{2.5}$$

A *finite classical group* is any group G satisfying $\Omega \leqslant G \leqslant A$ or $\overline{\Omega} \leqslant G \leqslant \overline{A}$ in one of the four cases $\mathbf{L}, \mathbf{S}, \mathbf{O}$ or $\mathbf{U}$. If G is such a classical group, then we call G a *linear group, symplectic group, orthogonal group* or *unitary group,* respectively. The convention for the field $\mathbf{F}$ is

$$\mathbf{F} = \mathbf{F}_{q^u}, \text{ where } q = p^f \text{ and } u = \begin{cases} 1 & \text{in cases } \mathbf{L}, \mathbf{S} \text{ and } \mathbf{O} \\ 2 & \text{in case } \mathbf{U}. \end{cases}$$

We also denote by α the automorphism of $\mathbf{F}$ given by $\lambda^\alpha = \lambda^q$ for $\lambda \in \mathbf{F}$. We write

$$L_n^{\pm}(q) = PSL_n^{\pm}(q) = \begin{cases} L_n(q) & \text{if the sign is } + \\ U_n(q) & \text{if the sign is } -. \end{cases}$$

Assume $\mathbf{F}, \kappa, \mathbf{f}, \mathbf{Q}, \sigma$ and τ as in above discussion, and let $X \in \{\Omega, S, I, \Lambda, \Xi, A\}$. We shall call $(V, \mathbf{F}, \kappa)$ or briefly, (V, κ), a *classical geometry.* If case $\mathbf{L}, \mathbf{S}, \mathbf{O}$ or $\mathbf{U}$ holds then we call $(V, \mathbf{F}, \kappa)$ a *linear, symplectic, orthogonal* or *unitary geometry,* respectively. If W is a subspace of V, and κ_W is the restriction of κ to W ($W \times W$) such that κ is either non-degenerate or identically zero, then $(W, \mathbf{F}, \kappa_{\mathbf{W}})$ is also a classical geometry and we call it a *sub-geometry* of $(V, \mathbf{F}, \kappa)$. W is said to be *non-degenerate* if κ_W is non-degenerate and W is *totally singular* if $\kappa_W = 0$. We also define W to be *totally isotropic* if $\mathbf{f}_W = 0$. Let $v \in V \setminus 0$. We say that v is *singular* or *isotropic* if $\mathbf{f}(v, v) = 0$ in cases $\mathbf{L}, \mathbf{S}$, and $\mathbf{U}$. In case $\mathbf{O}$, we say that v is *isotropic* if $\mathbf{f}(v, v) = 0$, and *singular* if $Q(v) = 0$. Vectors which are not singular are called *non-singular* and those which are not isotropic will be called *non-isotropic.* We now state the Witt's Lemma concerning the classical geometries.

Proposition 2.1 (Witt's Lemma, Proposition 2.1.6[29]). *Assume that $(V_1, \kappa_1), (V_2, \kappa_2)$ are two isometric classical geometries and that W_i is a subspace of V_i for $i = 1, 2$. Further assume that there is an isometry g from (W_1, κ_1) to (W_2, κ_2). Then g extends to an isometry from (V_1, κ_1) to (V_2, κ_2).*

Corollary 2.2 (Corollary 2.1.7[29]). *All maximal totally singular subspaces of (V, κ) have the same dimension. And if κ is non-degenerate, this dimension is at most $\frac{n}{2}$.*

Let $(V, \mathbf{F}, \kappa)$ be a classical geometry with $\dim_{\mathbf{F}}(V) = n$ and G be an irreducible subgroup of $GL(V, \mathbf{F})$. For any field extension $\mathbf{E}$ of $\mathbf{F}$, we form the tensor product $V \otimes \mathbf{E}$, an n-dimensional space over $\mathbf{E}$. Let G act naturally on $V \otimes \mathbf{E}$ via $(v \otimes \lambda)g = vg \otimes \lambda$, $(v \in V, g \in G, \lambda \in \mathbf{E})$, and we may regard $G \leqslant GL(V \otimes \mathbf{E})$. We say that G is absolutely irreducible in $GL(V, \mathbf{F})$ if G remains irreducible in $GL(V \otimes \mathbf{E}, \mathbf{E})$ for all extension field $\mathbf{E}$ of $\mathbf{F}$. A subgroup of $PGL(V, \mathbf{F})$ is said to be absolutely irreducible if its pre-image in $GL(V, \mathbf{F})$ is absolutely irreducible.

For $\lambda \in \mathbf{F}$, define

$$V_\lambda = \begin{cases} \{v \in V \setminus 0 \mid \mathbf{f}(v, v) = \lambda\} & \text{in cases } \mathbf{L}, \mathbf{S}, \mathbf{U}; \\ \{v \in V \setminus 0 \mid Q(v) = \lambda\} & \text{in cases } \mathbf{O}. \end{cases} \tag{2.6}$$

The following lemma describes the orbits of Ω on V. Write Ω for $\Omega(V, \mathbf{F}, \kappa)$.

Lemma 2.3 ([29], Lemma 2.10.5). *Assume that* $n \geqslant 2$.

(i) In cases $\mathbf{L}$ *and* $\mathbf{S}$, *the group* Ω *is transitive on* $V \setminus 0$.

(ii) Suppose that either case $\mathbf{U}$ *holds with* $n \geqslant 3$, *or case* $\mathbf{O}$ *holds with* $n \geqslant 4$. *Then* Ω *is transitive on* V_λ *for any* λ.

(iii) $SU_2(q)$ *has* $q+1$ *orbits on* V_0 *(each of size* $q^2 - 1$*), and is transitive on* V_λ *for* $\lambda \neq 0$.

(iv) $\Omega_3(q)$ *has two orbits on* V_0 *(each of size* $\frac{1}{2}(q^2 - 1)$*), and is transitive on* V_λ *for* $\lambda \neq 0$.

(v) $\Omega_2^+(q)$ *has* $2(2, q-1)$ *orbits on* V_0, *and* $\Omega_2^\pm(q)$ *has* $(2, q-1)$ *orbits on* V_λ *for* $\lambda \neq 0$.

Let $(V, \mathbf{F}, Q)$ be a classical orthogonal geometry, where $\mathbf{F} := \mathbf{F}_q$, and Q is a quadratic form on V. Let $\mathbf{f} = \mathbf{f}_Q$, the associated bilinear form. Then $\mathbf{f}$ is a symmetric bilinear form on V. Write (v, w) for $\mathbf{f}(v, w)$. Then for any $v \in V$, we have $2Q(v) = (v, v)$. Suppose that q is odd then $Q(v) = \frac{1}{2}(v, v)$. Sometimes, it will be convenient to work with the standard basis for L, defined as follows:

Definition 2.4 ([29], Proposition 2.5.3). *The space* $(V, \mathbf{F}, Q)$ *has a basis of one of the following forms:*

(i) $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}, n = 2m,$ *where* $Q(e_i) = Q(f_i) = 0$ *and* $(e_i, f_j) = \delta_{ij}$, $\forall i, j$;

(ii) $\{e_1, \ldots, e_{m-1}, f_1, \ldots, f_{m-1}, x, y\}, n = 2m,$ *where* $Q(e_i) = Q(f_i) = 0$, $(e_i, f_j) = \delta_{ij}$ *and* $(e_i, x) = (e_i, y) = (f_i, x) = (f_i, y) = 0$, *for all* i, j, *and* $\{x, y\}$ *satisfying the following condition* $(Q(x), Q(y), (x, y)) = (1, \zeta, 1)$, *where* $t^2 + t + \zeta$ *is irreducible over* $\mathbf{F}$.

(iii) $\{e_1, \ldots, e_m, f_1, \ldots, f_m, x\}, n = 2m + 1,$ *where* $Q(e_i) = Q(f_i) = 0$, $(e_i, f_j) = \delta_{ij}$ *and* $(e_i, x) = (f_i, x) = 0$, *for all* i, j, *and* x *is non-singular.*

Define

$$sgn(Q) = \begin{cases} \circ & \text{if } n \text{ is odd} \\ + & \text{if } n \text{ is even and } (V, Q) \text{ has a basis of type 2.4(i)} \\ - & \text{if } n \text{ is even and } (V, Q) \text{ has a basis of type 2.4(ii)} \end{cases} \tag{2.7}$$

We divide case $\mathbf{O}$ into cases $\mathbf{O}^{\circ}, \mathbf{O}^{+}$ and $\mathbf{O}^{-}$, accordingly.

Proposition 2.5 ([29], Proposition 2.5.4). *For each* n, *there are precisely two isometry classes of orthogonal geometries in dimension* n.

(i) If n *is even, then the two isometry types are distinguished by the dimension of their maximal totally singular subspaces. Indeed, the maximal totally singular subspaces have dimension* $\frac{n}{2}$ *or* $\frac{n}{2} - 1$ *according as* $sgn(Q) = +$ *or* $sgn(Q) = -$.

(ii) If n *is odd, then the two isometry types are distinguished by the value of* $Q(x)$ *modulo* $(\mathbf{F}^*)^{\mathbf{2}}$, *where* x *is given in Definition 2.4(iii). The two geometries are similar, and all maximal totally singular subspaces have dimension* $\frac{1}{2}(n - 1)$.

Let β be a basis of V, the *discriminant* $D(Q)$ of Q, is

$$D(Q) \equiv det(f_\beta)(\text{mod}(\mathbf{F}^*)^2) \in \mathbf{F}^*/(\mathbf{F}^*)^{\mathbf{2}}.$$

We write $D(Q) = \square$ or $\boxtimes$, according as $D(Q)$ is a square or a non-square. When q is odd and n is even, the next result determines $sgn(Q)$.

Proposition 2.6 ([29], Proposition 2.5.10, 2.5.12, 2.5.13). *Assume that q is odd and $n = 2m$ is even. We have:*

(i) If $sgn(Q) = +$, then $D(Q) = \square$ if and only if $\frac{1}{2}m(q-1)$ is even;

(ii) If $sgn(Q) = -$, then $D(Q) = \square$ if and only if $\frac{1}{2}m(q-1)$ is odd;

(iii) $D(Q) = \square$ if and only if $sgn(Q) = (-)^{\frac{1}{2}(q-1)m}$.

(iv) V has a basis β such that f_β is either I_n or $diag(\lambda, 1, \cdots, 1)$, according as $D(Q) = \square$ or $D(Q) = \boxtimes$, where λ is a generator of $\mathbf{F}^*$.

Proposition 2.7 ([29], Propposition 2.5.11). *Assume that $V = V_1 \perp \cdots \perp V_t$, where each V_i is a non-degenerate subspace of V.*

(i) $D(Q) = \prod_{i=1}^{t} D(Q_{V_i})$;

(ii) If $dim(V_i)$ is even for all i, then $sgn(Q) = \prod_{i=1}^{t} sgn(Q_{V_i})$.

Proposition 2.8 ([29], Proposition 2.6.1). *Let $(V, \mathbf{F}, Q)$ be a classical orthogonal geometry in odd dimension $n = 2m+1$. Then there exists a basis β of $(V, \mathbf{F}, Q)$ such that $\mathbf{f}_\beta = \lambda \mathbf{I}_n$, where $D \equiv \lambda (mod\ (\mathbf{F}^*)^{\mathbf{2}})$.*

Let U be a non-degenerate subspace of V. Write $D(U) = D(Q_U)$ and $sgn(U) = sgn(Q_U)$. Let $(V, \mathbf{F}, Q)$ be a classical orthogonal geometry of type $\varepsilon \in \{\circ, \pm\}$ with q odd and $dimV = n$. For any non-zero vector x in V, a one-space with representative x will be called a *point* in V and denoted by $\langle x \rangle$. We now define a type function $\rho = \rho_V$ on $V \setminus \{0\}$ as follows: If x is a singular vector in V, that is, $x \in V \setminus \{0\}$ and $Q(x) = 0$, then $\rho(x) = 0$. If $n = \mathrm{dim}V = 2m$ is even, then $\rho(x) = Q(x)$. If $n = 2m+1$ is odd, then $\rho(x) = \mathrm{sgn}(x^\perp)$. Assume that $dimV$ is odd. Let x be a non-singular vector in V. We say that x is a *plus vector* if $\rho(x) = +$; and x is a *minus vector* if $\rho(x) = -$. We also say point $\langle x \rangle$ is of plus or minus type according to whether its representative x is a plus or minus vector. Let $x \in V$ be a non-singular vector with $\rho(x) = \xi$. Define $\mathfrak{E}_\xi^\varepsilon(V)$ to be the set of all non-singular points of type ξ in V. We can omit either the index ξ or the vector space V, when they are understood. From definition, we have

$$\mathfrak{E}^{\varepsilon}_{\xi}(V) = \{\langle v\rangle \subseteq V \mid Q(v) \neq 0 \text{ and } \rho(v) = \xi\}.$$

Recall that $V_\gamma = \{v \in V \setminus 0 \mid Q(v) = \gamma\}$, for any $\gamma \in \mathbf{F}$.

Remark 2.9 *If dimV is even then* $\mathfrak{E}^{\varepsilon}_{\xi}(V) = \{\langle v\rangle \mid v \in V_\xi\}$. *However, if dimV is odd, they may be not equal.*

Lemma 2.10 *Assume that dimV is odd. Let x, y be two non-singular vectors V. Then two vectors x, y (or two non-singular points $\langle x\rangle, \langle y\rangle$) have the same type if and only if* $Q(x) \equiv Q(y)$ *(mod* $(\mathbf{F}^*)^2$*). Thus if x is a non-singular vector in V with $Q(x) = \gamma \neq 0$ and* $\rho(x) = sgn(x^\perp) = \xi$. *Then* $\mathfrak{E}^{\varepsilon}_{\xi}(V) = \bigcup_{\gamma \in \xi \mathbf{F}^{*2}} \{\langle v\rangle \mid v \in V_\gamma\}$. *In particular, if $q = 3$ then* $\mathfrak{E}_{\xi}(V) = \{\langle v\rangle \mid v \in V_\gamma\}$.

Proof. Assume first that $Q(x) \equiv Q(y)$ (mod $(\mathbf{F}^*)^2$). According to Proposition 2.5(ii), $\langle x\rangle$ and $\langle y\rangle$ are isometric. By Witt's Lemma, this isometry extends to an isometry g of V such that $\langle x\rangle g = \langle y\rangle$. As $\langle x\rangle, \langle y\rangle$ are non-degenerate, $x^\perp g = y^\perp$. It follows that $x^\perp$ and $y^\perp$ are isometric, and hence $sgn(x^\perp) = sgn(y^\perp)$, so that $\rho(x) = \rho(y)$. Now, assume that x, y have the same type. By Witt's Lemma and Proposition 2.5(i), there exists an isometry between $x^\perp$ and $y^\perp$. This isometry can extend to an isometry g of V such that $(x^\perp)g = y^\perp$. As $(x^\perp)^\perp = \langle x\rangle$, and $(y^\perp)^\perp = \langle y\rangle$, $(\langle x\rangle)g = \langle y\rangle$. Thus $xg = \mu y$ for some $\mu \in \mathbf{F}^*$. Therefore $Q(x) = Q(xg) = Q(\mu y) = \mu^2 Q(y)$. If $\mathbf{F} = \mathbf{F}_3$, then $\mathbf{F}^{*2} = \{1\}$. The result follows. ∎

When n is odd, for any non-zero vector x in V, we denote by $S(n, x)$ the number of all vectors $v \in V$ with $\rho(v) = \rho(x) = \xi \in \{0, \pm\}$, and $Q(v) = Q(x)$. When n is even, for any $\gamma \in \mathbf{F}$, set $S^\varepsilon(n, \gamma) = |V_\gamma|$, the number of solutions to $Q(v) = \gamma$ in V.

Lemma 2.11 *Assume q is odd, x is a non-zero vector in V with $\rho(x) = \xi \in \{0, \pm\}$, and $\gamma \in \mathbf{F}$.*

1. $S^{\varepsilon}(2k,\gamma) = \begin{cases} q^{2k-1}+\varepsilon(q^k - q^{k-1}) & \textit{if } \gamma = 0, \\ q^{2k-1} - \varepsilon q^{k-1} & \textit{if } \gamma \neq 0; \end{cases}$
2. $S(2k+1,x) = q^{2k} + \xi q^k$.

Proof. (1) follows from Proposition 9.10 in [14]. For (2), let $\gamma = Q(x)$. Firstly, assume that x is a non-singular vector. Then $V = \langle x \rangle \perp x^{\perp}, \mathrm{sgn} x^{\perp} = \rho(x) = \xi$ and $\mathrm{dim} x^{\perp} = 2k$. According to Lemma 2.10, for any $v \in V$, if $Q(v) = Q(x)$ then $\rho(v) = \rho(x)$. Thus $S(2k+1,x) = |V_{\gamma}|$. Now, for any $v \in V$ with $Q(v) = \gamma$, write $v = \varphi x + v_0$, where $\varphi \in \mathbf{F}$ and $v_0 \in x^{\perp}$. Then $Q(v_0) = Q(v) - \varphi^2 Q(x) = \gamma(1-\varphi^2)$. If $\varphi = \pm 1$, then $Q(v_0) = 0$, hence by (1), there are $2S^{\xi}(2k,0) = 2(q^{2k-1} + \xi(q^k - q^{k-1}))$ such v. If $\varphi \neq \pm 1$, then $Q(v_0) = \gamma(1-\varphi^2) \neq 0$, hence by (1), again, there are $(q-2)S^{\xi}(2k,\gamma(1-\varphi^2)) = (q-2)(q^{2k-1} - \xi q^{k-1}))$ such v. Thus, $S(2k+1,x) = 2S^{\xi}(2k,0) + (q-2)S^{\xi}(2k,\gamma(1-\varphi^2)) = q^{2k} + \xi q^k$. Finally, assume that x is a singular vector. Then $S(2k+1,x) = |V_0|$. Observe that V always contains a non-singular vector y. Let $\eta = \rho(y)$ and $\mu = Q(y)$. Arguing as previous case, we have $V = \langle y \rangle \perp y^{\perp}$, $\mathrm{sgn}(y^{\perp}) = \eta, Q(y) = \mu \in \mathbf{F}^*$ and $\mathrm{dim} y^{\perp} = 2k$. For any $v \in V$ with $Q(v) = 0$, write $v = \varphi y + v_0$, where $\varphi \in \mathbf{F}$ and $v_0 \in y^{\perp}$. Then $Q(v_0) = Q(v) - \varphi^2 Q(y) = -\mu\varphi^2$. If $\varphi = 0$, then $Q(v_0) = 0$, hence by (1), there are $S^{\eta}(2k,0) = q^{2k-1} + \eta(q^k - q^{k-1})$ such v. If $\varphi \neq 0$, then $Q(v_0) = -\mu\varphi^2 \neq 0$, hence by (1), again, there are $(q-1)S^{\eta}(2k,-\mu\varphi^2) = (q-1)(q^{2k-1} - \eta q^{k-1}))$ such v. Thus, $S(2k+1,x) = S^{\eta}(2k,0) + (q-1)S^{\eta}(2k,-\mu\varphi^2) = q^{2k}$. ■

2.2 Definitions and Structures of classes $\mathcal{C}$ and $\mathcal{S}$

Let G_0 be one of the finite classical groups

$$PSL_n(q), PSU_n(q), PSp_n(q)(\ n \text{ even }), P\Omega_n^{\pm}(q)(\ n \text{ even}), \Omega_n(q)(\ nq \text{ odd}). \tag{2.8}$$

Table 2.1: Rough description of classes $\mathcal{C}_i$

$\mathcal{C}_i$	rough description	rough structure in $GL_n(q)$
$\mathcal{C}_1$	stabilizers of totally	maximal parabolic
	singular or non-singular subspaces	
$\mathcal{C}_2$	stabilizers of decompositions $V = \bigoplus_{i=1}^{t} V_i$, dim$(V_i)$=$a$	$GL_a(q) \wr S_t, n = at$
$\mathcal{C}_3$	stabilizers of extension fields of $\mathbf{F}_q$ of prime index b	$GL_a(q^b) \cdot b, n = ab$, b prime
$\mathcal{C}_4$	stabilizers of tensor product decompositions $V = V_1 \bigotimes V_2$	$GL_a(q) \circ GL_b(q), n = ab$
$\mathcal{C}_5$	stabilizers of subfields of $\mathbf{F}_q$ of prime index b	$GL_n(q_0), q = q_0^b, b$ prime
$\mathcal{C}_6$	normalizers of sympletic-type r-groups (r prime) in absolutely irreducible representations	$(\mathbb{Z}_{q-1} \circ r^{1+2a}) \cdot Sp_{2a}(r), n = r^a$
$\mathcal{C}_7$	stabilizers of decopositions $V = \bigotimes_{i=1}^{t} V_i$, dim$(V_i)$=$a$	$\overbrace{(GL_a(q) \circ \cdots \circ GL_a(q))}^{t} \cdot S_t$
$\mathcal{C}_8$	classical subgroups	$Sp_n(q), n$ even $O_n^{\varepsilon}(q), q$ odd $GU_n(\sqrt{q}), q$ a square

Let V be the natural n-dimensional vector space over the finite field $\mathbf{F}$ associated with G_0, and let Ξ be the full semilinear classical group corresponding to G_0. M. Aschbacher defined eight collections $\mathcal{C}_i(\Xi), (i = 1, \ldots, 8)$ of natural subgroups of Ξ. The rough description of the subgroups $H \cap GL_n(q)$, where $H \in \mathcal{C}_i(\Xi)$ and $G_0 = PSL_n(q)$ is given in Table 2.1. For each group X satisfying either $\Omega \leqslant X \leqslant A$ or $\overline{\Omega} \leqslant X \leqslant \overline{A}$, the collection $\mathcal{C}(X)$ is a union of families $\mathcal{C}_i(X)$, where $i = 1, \cdots, 8$, we also define $\mathcal{C}_i = \cup\mathcal{C}_i(X)$ and $\mathcal{C} = \cup\mathcal{C}(X)$. In each collection $\mathcal{C}_i$, there are many sub-collections called *type.* If $\mathbf{T}$ is a type in $\mathcal{C}_i$, and H belongs to $\mathbf{T}$, then we say that H *is of type* $\mathbf{T}$. Moreover, for any group $H_\Xi \in \mathcal{C}(\Xi)$, there are corresponding group $H_X = H_\Xi \cap X \in \mathcal{C}_X$, for any X as above. If Y satisfies the same condition as X, then H_X is called X-*associate* of H_Y. Now we define the class $\mathcal{S}$ of subgroups of G, where $G_0 \trianglelefteq G \leqslant Aut(G_0)$ and G_0 as in (2.8).

Definition 2.12 *The subgroup H of G lies in $\mathcal{S}$ if and only if the following hold.*

(a) *The socle S of H is a non-abelian simple group (i.e., H is almost simple).*

(b) *If L is the full covering group of S, and if $\rho : L \to GL(V)$ is a representation of L such that $\overline{\rho(L)} = S$, then ρ is absolutely irreducible.*

(c) *$\rho(L)$ cannot be realized over a proper subfield of $\mathbf{F}$.*

(d) *If $\rho(L)$ fixes a non-degenerate quadratic form on V, then $G_0 = P\Omega_n^{\varepsilon}(q)$.*

(e) *If $\rho(L)$ fixes a non-degenerate symplectic form on V, but no non-degenerate quadratic form, then $G_0 = PSp_n(q)$.*

(f) *If $\rho(L)$ fixes a non-degenerate unitary form on V, then $G_0 = PSU_n(q)$.*

(g) *If $\rho(L)$ does not satisfy the conditions in $(d), (e)$ or (f), then $G_0 = PSL_n(q)$.*

We now state the subgroup structure theorem due to Aschbacher.

Theorem 2.13 *Let G be a group such that $G_0 \trianglelefteq G \leqslant \overline{\Xi}$, with G_0 and Ξ as in (2.8) above, and let H be a subgroup of G not containing G_0. Then either H is contained in a member of $\mathcal{C}(G)$ or $H \in \mathcal{S}$.*

Kleidman and Liebeck have determined the maximality of subgroups of $\mathcal{C}(G)$ when $n \geqslant 13$. Here is the main theorem of [29].

Theorem 2.14 *Let G_0 be as in (2.8) and let $G_0 \trianglelefteq G \leqslant Aut(G_0)$. Then*

(A) *the group-theoretic structure of each $H \in \mathcal{C}(G)$ is known;*

(B) *the conjugacy amongst the members of $\mathcal{C}(G)$ is known;*

(C) *for $H \in \mathcal{C}(G)$ all overgroups of H which lie in $\mathcal{C}(G) \bigcup \mathcal{S}$ are known (for $n \geqslant 13$).*

In order to determine the structure and conjugacy amongst members of $\mathcal{C}$, we need the following definitions. For any group X, Y satisfying $Y \leqslant \overline{\Omega} \leqslant X \leqslant \overline{A}$, define $[Y]_X$ as the set of X-conjugates of Y. Write $[Y] = [Y]_{\overline{\Omega}}$ and define $[Y]^X$ to be the set of $\overline{\Omega}$-classes contained in the X-class $[Y]_X$. Thus $[Y]^X = \{[Y^x] \mid x \in X\}$. Define $e = |[H_{\overline{\Omega}}]^{\overline{A}}|$, (this is parameter c in [29]) and write $[H_{\overline{\Omega}}]^{\overline{A}} = \{[H_1], \cdots, [H_e]\}$. Also define $X_i = X_{[H_i]} =$

Table 2.2: The simple classical groups

L	Lie notation Lie rank ℓ	d	$\lvert Out(L)\rvert$	$\lvert L\rvert$
$L_n(q)$	$A_{n-1}(q)$ $n-1$	$(n, q-1)$	$2df, n \geqslant 3$ $df, n=2$	$\frac{1}{d}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-1)$
$U_n(q)$	${}^2A_{n-1}(q)$ $[\frac{n}{2}]$	$(n, q+1)$	$2df, n \geqslant 3$ $df, n=2$	$\frac{1}{d}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i)$
$PSp_{2n}(q)$	$C_n(q)$ n	$(2, q-1)$	$df, n \geqslant 3$ $2f, n=2$	$\frac{1}{d}q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$
$\Omega_{2n+1}(q)$ q odd	$B_n(q)$ n	2	$2f$	$\frac{1}{d}q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$
$P\Omega_{2n}^{+}(q)$ $n \geqslant 3$	$D_n(q)$ n	$(4, q^n-1)$	$2df, n \neq 4$ $6df, n=4$	$\frac{1}{d}q^{n(n-1)}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$
$P\Omega_{2n}^{-}(q)$ $n \geqslant 2$	${}^2D_n(q)$ $n-1$	$(4, q^n+1)$	$2df$	$\frac{1}{d}q^{n(n-1)}(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)$

$N_X(H_i)\overline{\Omega}$. Then $\ddot{X}_i = N_X(H_i)\overline{\Omega}/\overline{\Omega}$. For each $i = 1, \cdots, e$, write $H_{G,i}$ for G-associate of H_i. The action of $\ddot{A}$ on $[H_{\overline{\Omega}}]^{\overline{A}}$ induces a homomorphism $\pi = \pi_{H_{\overline{\Omega}}}$ from $\ddot{A}$ to the symmetric group S_e.

2.3 Properties of finite simple groups

By the theorem of classification of finite simple groups, every non-abelian finite simple group is either an alternating group $A_n, n \geqslant 5$, a finite simple group of Lie type, or one of the 26 sporadic groups. The finite simple groups of Lie type is divided into the classical groups and the exceptional groups. The orders of alternating group A_n is $\frac{1}{2}n!$. The orders of remaining simple groups are given in Tables 2.2, 2.3 and 2.4.

Theorem 2.15 *The excetional groups in Table 2.3 are simple except for*

${}^2B_2(2) \cong \mathbb{Z}_5 : \mathbb{Z}_4$

$G_2(2) \cong Aut(U_3(3)) \cong U_3(3).2$

${}^2G_2(3) \cong Aut(L_2(8)) \cong L_2(8).3$

${}^2F_4(2) \cong {}^2F_4(2)'.2.$

Table 2.3: The simple exceptional groups

L	ℓ	d	$\lvert Out(L)\rvert$	$\lvert L\rvert$
$G_2(q)$	2	1	f if $p \neq 3$ $2f$ if $p = 3$	$q^6(q^2-1)(q^6-1)$
$F_4(q)$	4	1	$(2,p)f$	$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
$E_6(q)$	6	$(3,q-1)$	$2df$	$\frac{1}{d}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-1)$
$E_7(q)$	7	$(2,q-1)$	df	$\frac{1}{d}q^{63}\prod_{i\in\{2,6,8,10,12,14,18\}}(q^i-1)$
$E_8(q)$	8	1	f	$q^{120}\prod_{i\in\{2,8,12,14,18,20,24,30\}}(q^i-1)$
${}^2B_2(q), q = 2^{2m+1}$	1	1	f	$q^2(q^2+1)(q-1)$
${}^2G_2(q), q = 3^{2m+1}$	1	1	f	$q^3(q^3+1)(q-1)$
${}^2F_4(q), q = 2^{2m+1}$	2	1	f	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$
${}^3D_4(q)$	2	1	$3f$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$
${}^2E_6(q)$	4	$(3,q+1)$	$2df$	$\frac{1}{d}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-(-1)^i)$

The group ${}^2F_4(2)'$ is simple and is called the Tits group.

We define $Lie(p)$ to be the set of *simple* groups of Lie type over fields of characteristic p. We include $PSp_4(2)'$ and ${}^2F_4(2)'$ in $Lie(2)$. We also define $Lie(p') = \cup_{r\neq p}Lie(r)$. A member of $Lie(p)$ is said to be *twisted* if it lies in one of the families ${}^2A_\ell, {}^2B_2, {}^2D_\ell, {}^3D_4, {}^2E_6, {}^2F_4$ or ${}^2G_2$. Otherwise, L is *untwisted.*

Theorem 2.16 *The isomorphisms among the groups in Tables 2.2, 2.3 and 2.4 and the alternating groups are precisely those given in Theorem 2.15 together with the following isomorphisms:*

(i) $SL_2(q) \cong Sp_2(q) \cong SU_2(q)$.

(ii) For q odd, $L_2(q) \cong \Omega_3(q)$.

(iii) $O_2^\pm(q) \cong D_{2(q\mp1)}, SO_2^\pm(q) \cong \mathbb{Z}_{q\mp1}.(2,q)$ *and* $\Omega_2^\pm(q) \cong \mathbb{Z}_{q\mp1/(2,q)}$.

(iv) $\Omega_4^+(q) \cong SL_2(q) \circ SL_2(q)$.

(v) $\Omega_4^-(q) \cong L_2(q^2)$.

(vi) For q odd, $PSp_4(q) \cong \Omega_5(q)$.

(vii) $P\Omega_6^\pm(q) \cong L_4^\pm(q)$.

(viii) $L_2(2) \cong S_3$.

Table 2.4: The simple sporadic groups

L	d	$\|Out(L)\|$	$\|L\|$
M_{111}	1	1	$2^4.3^2.5.11$
M_{12}	2	2	$2^6.3^3.5.11$
M_{22}	12	2	$2^7.3^2.5.7.11$
M_{23}	1	1	$2^7.3^2.5.7.11.23$
M_{24}	1	1	$2^{10}.3^3.5.7.11.23$
J_1	1	1	$2^3.3.5.7.11.19$
J_2	2	2	$2^7.3^3.5^2.7$
J_3	3	2	$2^7.3^5.5.17.19$
J_4	1	1	$2^{21}.3^3.5.7.11^3.23.29.31.37.43$
HS	2	2	$2^9.3^2.5^3.7.11$
Suz	6	2	$2^{13}.3^7.5^2.7.11.13$
McL	3	2	$2^7.3^6.5^3.7.11$
Ru	2	1	$2^{14}.3^3.5^3.7.13.29$
$He = F_7$	1	2	$2^{10}.3^3.5^2.7^3.17$
Ly	1	1	$2^8.3^7.5^6.7.11.31.37.67$
$O'N$	3	2	$2^9.3^4.5.7^3.11.19.31$
Co_1	2	1	$2^{21}.3^9.5^4.7^2.11.13, 23$
Co_2	1	1	$2^{18}.3^6.5^3.7.11.23$
Co_3	1	1	$2^{17}.3^7.5^3.7.11.23$
Fi_{22}	6	2	$2^{17}.3^9.5^2.7.11.13$
Fi_{23}	1	1	$2^{18}.3^{13}.5^2.7.11.13.17.23$
Fi'_{24}	3	2	$2^{21}.3^{16}.5^2.7^3.11.13.17.23.29$
$HN = F_5$	1	2	$2^{14}.3^6.5^6.7.11.19$
$Th = F_3$	1	1	$2^{15}.3^{10}.5^3.7^2.13.19.31$
$BM = F_2$	2	1	$2^{41}.3^{13}.5^6.7^2.11.13.17.19.23.31.47$
$M = F_1$	1	1	$2^{46}.3^{20}.5^9.7^6.11^2\times$ $13^3.17.19.23.29.31.41.47.59.71$

(ix) $L_2(3) \cong A_4$.

(x) $L_2(4) \cong L_2(5) \cong A_5$.

(xi) $L_2(7) \cong L_3(2)$.

(xii) $L_2(9) \cong A_6$.

(xiii) $L_4(2) \cong A_8$.

(xiv) $U_3(2) \cong 3^2.Q_8$.

(xv) $U_4(2) \cong PSp_4(3)$.

(xvi) $Sp_4(2) \cong S_6$

Let L be a simple group. Denote by $Out(L)$ the outer automorphism group of L, which is $Aut(L)/L$. The orders of $Out(L)$ are given in Tables 2.2, 2.3 and 2.4 for simple groups of Lie type and sporadic groups. The orders of $Out(A_n), n \geqslant 5$ are given in the following theorem.

Theorem 2.17 (Theorem 5.1.3[29].) *If $n \geqslant 5$ and $n \neq 6$, then $Aut(A_n) = S_n$ and hence $|Out(A_n)| = 2$. However $|Out(A_6)| = 4$.*

Lemma 2.18 (Lemma 2.1[47]). *Let $2 \leqslant a_1 < \cdots < a_\ell$ be integers, $\varepsilon_1, \ldots, \varepsilon_\ell \in \{\pm 1\}$. Then*

$$\frac{1}{2} < \frac{(q^{a_1} + \varepsilon_1) \ldots (q^{a_\ell} + \varepsilon_\ell)}{q^{a_1 + \cdots + a_\ell}} < 2.$$

Using the above lemma, we can easily verify the following:

Lemma 2.19 *Assume that L is a finite simple group of Lie type. Then $|Aut(L)| \leqslant f(L)$, where $f(L)$ is given in Table 2.5.*

Primitive prime divisors

Let q and n be integers with $q \geqslant 2$ and $n \geqslant 3$. A prime s is a *primitive prime divisor* of $q^n - 1$, denoted by q_n, if $s | q^n - 1$ but s does not divide $q^i - 1$ for $i < n$. The Zsigmondy's theorem (Zsigmondy [48]) asserted the existence of primitive prime divisor.

Table 2.5: Upper bounds for order of $Aut(L)$.

L	$f(L)$	L	$f(L)$
$L_n(q), n \geqslant 3$	q^{n^2}	$E_8(q)$	q^{249}
$PSp_{2n}(q)$	q^{2n^2+n+1}	$F_4(q)$	q^{53}
$U_n(q), n \geqslant 3$	q^{n^2}	${}^2E_6(q)$	q^{79}
$P\Omega_{2n}^+(q), n \neq 4$	q^{2n^2-n+1}	$G_2(q)$	q^{15}
$P\Omega_8^+(q)$	$3q^{29}$	${}^3D_4(q)$	$3q^{29}$
$P\Omega_{2n}^-(q)$	q^{2n^2-2n+3}	${}^2F_4(q)$	q^{27}
$\Omega_{2n+1}(q)$	q^{2n^2+n+1}	$Sz(q)$	q^6
$E_6(q)$	q^{79}	${}^2G_2(q)$	q^8
$E_7(q)$	q^{134}	$P\Omega_{2n}^-(q)$	$2q^{2n^2-2n+2}$

Theorem 2.20 *There exists a primitive prime divisor of $q^n - 1$, provided $q \geqslant 2, n \geqslant 3$ and $(q, n) \neq (2, 6)$.*

When n is odd and $(q, n) \neq (2, 3)$, then there is a primitive prime divisor q_{2n} of $q^{2n} - 1$, and we write $q_{-n} = q_{2n}$.

Representations of simple groups

Let G be a finite group and P, M be subgroups of G. Denote by $M\backslash G/P$ a set of representatives for double cosets of G on P and M. Let $\mathfrak{C} = G/P$, the right cosets of G on P. Then M acts on $\mathfrak{C}$ by right multiplication.

Lemma 2.21 *Let M, P be subgroups of a finite group G. Then*
(i) M has $|M\backslash G/P|$ orbits on G/P, where M acts on G/P by right multiplication.
(ii) $(1_M^G, 1_P^G) = |M\backslash G/P|$.

Proof. (i) Suppose that $M\backslash G/P = \{x_1, \ldots, x_k\}$. Then $Px_iM \cap Px_jM = \emptyset$ if $i \neq j, i, j = 1, \ldots, k$. Clearly, Px_iM are distinct orbits of M on G/P. As $G = \cup_{i=1}^k Px_iM$, $\{Px_iM\}_{i=1}^k$ is a complete set of orbits of M on G/P.

(ii) By Mackey's formula ([23]),

$$(1_M^G, 1_P^G) = \sum_{t \in M \backslash G/P} (1^t_{M^t \cap P}, 1_{M^t \cap P}) = \sum_{t \in M \backslash G/P} 1 = |M \backslash G/P|.$$

This proves the lemma. ■

Let G be a group and p a prime. A *p-modular representation* of G is a homomorphism form G to $GL_n(\mathbf{F})$ for some n and some field $\mathbf{F}$ of characteristic p. A *projective p-modular representation* of G is a homomorphism from G to $PGL_n(\mathbf{F})$. If G has a faithful projective p modular representation of degree n, then G embeds in $PGL_n(\mathbf{F})$, written $G \preceq PGL_n(\mathbf{F})$. We define

$$\begin{aligned}
R_{\mathbf{F}}(G) &= min\{n \mid G \preceq PGL_n(\mathbf{F})\}, \\
R_p(G) &= min\{R_{\mathbf{F}}(G) \mid \mathbf{F} \text{ a field of characteristic } p\}, \\
R_{p'}(G) &= min\{R_s(G) \mid \ s \text{ prime}, s \neq p\}, \\
R(G) &= min\{R_p(G) \mid \ \text{all primes } p\}.
\end{aligned} \tag{2.9}$$

If $\overline{\mathbf{F}}_p$ is the algebraic closure of $\mathbf{F}_p$, then $R_p(G) = min\{n \mid G \preceq PGL_n(\overline{\mathbf{F}}_p)\}$.

For irreducible projective representations, we have similar definitions.

$$\begin{aligned}
R^i_{\mathbf{F}}(G) &= min\{n \mid G \text{ embeds irreducibly in } PGL_n(\mathbf{F})\} \\
R^i_p(G) &= min\{R^i_{\mathbf{F}}(G) \mid \mathbf{F} \text{ a field of characteristic } p\} \\
&= min\{n \mid G \text{ embeds irreducibly in } PGL_n(\overline{\mathbf{F}}_p)\} \\
R^i_{p'}(G) &= min\{R^i_s(G) \mid \ s \text{ prime}, s \neq p\}, \\
R^i(G) &= min\{R^i_p(G) \mid \ \text{all primes } p\}.
\end{aligned} \tag{2.10}$$

Embedding of finite simple groups

We first define the *Frobenius-Schur indicators* of the irreducible Brauer characters. Let G be a finite group, $\mathbf{F}$ be a field and M be a finitely generated, absolutely irreducible

$\mathbf{F}G$-module which affords the Brauer character χ. If M is self-dual, then M carries a non-degenerate G-invariant bilinear form, which is symmetric or alternating and unique up to scalar multiplication. If the characteristic of $\mathbf{F}$ is 2, and M is not the trivial module, then the invariant form is alternating. If the characteristic of $\mathbf{F}$ is odd and M carries a non-degenerate symmetric bilinear form, this form is the associated bilinear form of a quadratic form on M, which is non-degenerate and $G-$invariant. If M is not self-dual, it does not carry any non-trivial $G-$invariant bilinear form. The Frobenius-Schur indicator of M, written $\mathrm{ind}(M)$ or $\mathrm{ind}(\chi)$, is an integer of the set $\{-1, 0, +1\}$. It is defined to be 0 if and only if M is not self-dual. The Frobenius-Schur indicator of M is set to be $+1$ if and only if M carries non-degenerate $G-$invariant quadratic form. Thus the indicator of M is -1 if and only if M is self-dual and carries a non-degenerate $G-$invariant bilinear alternating form, but no $G-$invariant quadratic form. If $ind(\chi) = +1$, then G embeds in an orthogonal groups and the value of χ determine the smallest field $\mathbf{F}$ such that G embeds in $GO_n^{\varepsilon}(\mathbf{F})$ for some $\varepsilon \in \{+, -\}$, where $n = dim(M)$. If characteristic of $\mathbf{F}$ is 2 and n is even, then G embeds in $Sp_n(\mathbf{F})$. If $ind(\chi) = -1$, then G embeds into $Sp_n(\mathbf{F})$, where the field $\mathbf{F}$ is determined by the value of χ. If $ind(\chi) = 0$, then G may or may not embed in a suitable unitary groups.

2.4 Representations of Symmetric groups

We collect some information on representations of symmetric and alternating groups in characteristic p. We are interested in the lower bounds for absolutely irreducible p-modular representations of these groups. We begin by some basic definitions.

Definition 2.22 *(i) $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, written $\lambda \vdash n$, provided $\lambda_i, i = 1, \ldots, k$ are integers, with $\lambda_1 \geqslant \ldots \geqslant \lambda_k > 0$ and $\sum_{i=1}^{k} \lambda_i = n$. We often gather the equal parts together and write $\lambda = (\lambda_1^{a_1}, \cdots, \lambda_h^{a_h})$, where $\lambda_1 > \ldots \lambda_h > 0$ and $\sum_{i=1}^{h} a_i\lambda_i = n$.*
(ii) For any prime p, a partition $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \cdots, \lambda_h^{a_h})$ is called p-regular if $1 \leqslant a_i \leqslant p-1$

for all $i = 1, \dots, h$.

(iii) *Let* $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \cdots, \lambda_h^{a_h})$ *be a p-regular partition. Then* λ *is called a JS- partition if* $\lambda_i - \lambda_{i+1} + a_i + a_{i+1} \equiv 0 \ mod \ p$.

Next we construct the fully deleted module for alternating groups. This is the smallest faithful irreducible representation of A_n when n is at least 9. Let $n \geqslant 5$, let p be a prime, and let S_n act on the permutation module F_p^n by permuting the coordinates naturally. Define submodules U, W of F_p^n by

$$U = \{(a_1, a_2, \dots, a_n) | \sum_{i=1}^{n} a_i = 0\}, W = \{(a, a, \dots, a) | a \in F_p\}.$$

Let $V = U/(U \cap W)$, and

$$\varepsilon_p(n) = \begin{cases} 0 & \text{if } p \text{ does not divide n} \\ 1 & \text{if } p|n. \end{cases}$$

Then $dim(V) = n - 1 - \varepsilon_p(n)$. V is said to be the *fully deleted permutation module* for A_n over F_p. Define the natural symmetric bilinear form $f : U \times U \to F_p$ given by $f((a_1, \dots, a_n), (b_1, \dots, b_n)) = \sum_{i=1}^{n} a_i b_i$. A_n preserves this form and f induces a symmetric bilinear form on the fully deleted permutation module V, and we obtain the embedding $A_n \leqslant \Omega(V, F_p, f)$.

Proposition 2.23 ([25], Theorem 6). *Let* $n \geqslant 10$ *and* V *be a non-trivial irreducible* A_n*-module in characteristic* p*. Suppose that* $dim(V) \leqslant n$*. Then* V *is isomorphic to the fully deleted module for* A_n*.*

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_h)$ be a partition of n, where $\lambda_1 \geqslant \cdots \geqslant \lambda_h$ and $\lambda_1 + \cdots + \lambda_h = n$. The subgroup $S_\lambda = S_{\{1,2,\cdots,\lambda_1\}} \times S_{\{\lambda_1+1,\cdots,\lambda_1+\lambda_2\}} \times S_{\{\lambda_1+\cdots+\lambda_{h-1}+1,\cdots,\lambda_1+\cdots+\lambda_h\}}$ is called a *Young subgroup* of S_n associated to λ. Denote by M^λ the permutation module $1_{S_\lambda}^{S_n}$. Let

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash n$ be a partition of n. The *Young diagram* associated to λ, written $[\lambda]$, is the subset $(i,j) \in \{\mathbb{Z} \times \mathbb{Z} \mid 1 \leqslant i, 1 \leqslant j \leqslant \lambda_i\}$, which is consisting of n nodes. The λ-*tableaux* is one of the $n!$ arrays of integers obtained by replacing each node in $[\lambda]$ by one of the integers $1, 2, \cdots, n$, without repetition. The symmetric group S_n acts naturally on the set of λ-tableaux. Let t be a tableaux. The *row-stabilizer*, R_t, is a subgroup of S_n which fixes the rows of t. The *column stabilizer*, C_t, of t is defined similarly. Using the row stabilizer, we can define an equivalence relation on the set of all λ-tableaux by saying that $t_1 \sim t_2$ if and only if $t_1\pi = t_2$ for some $\pi \in R_{t_1}$. The *tabloid* $\{t\}$ containing t is the equivalence class of t under this equivalence relation. The group S_n acts on tabloids as follows: $\{t\}\pi = \{t\pi\}$ for any tabloid t and $\pi \in S_n$. Let $\mathbf{F}$ be any field. Then M^λ becomes an $\mathbf{F}S_n$-module via the action of S_n on λ-tabloids. Let t be a λ-tableau. The element $\kappa_t = \sum_{\pi \in C_t} (sgn\pi)\pi \in \mathbf{F}S_n$ is called the *signed column sum.* The *polytabloid*, e_t, associated with the tabloid t is $e_t = \{t\}\kappa_t$. The *Specht module* S^λ for the partition λ is the submodule of M^λ generated by polytabloids. We next find a basis for Specht module. Let t be a λ-tableaux. Then t is a *standard tableaux* if the numbers increase along the rows and down the columns of t. We say $\{t\}$ is a *standard tabloid* if there is a standard tableaux in the equivalence class $\{t\}$. Also e_t is a *standard polytabloid* if t is standard.

Theorem 2.24 ([24], 8.4) $\{e_t \mid t \text{ is a standard } \lambda\text{-tableaux}\}$ is a basis for S^λ.

The dimension of Specht modules can be given by the Hook formula. We need some definitions before we can give the formula. Let $[\lambda]$ be a Young diagram. The *conjugate diagram* $[\lambda']$ is obtained by interchanging the rows and columns in $[\lambda]$. If $\lambda = (\lambda_1, \lambda_2, \cdots)$ then $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$, where λ'_i is the length of i^{th} column in $[\lambda]$. The $(i,j) - $*hook* of $[\lambda]$ consists of the (i,j)-node along with the $\lambda_i - j$ nodes to the right of it and the $\lambda'_j - i$ nodes below it. The *length* of (i,j)-hook is $h_{ij} = \lambda_i + \lambda'_j + 1 - i - j$. The *hook graph* of $[\lambda]$ is obtained from the Young diagram $[\lambda]$ by replacing the (i,j)-node by the numbers h_{ij}.

Theorem 2.25 ([24], 20.1). The dimension of the Specht module S^λ is given by

$$dim S^\lambda = \frac{n!}{\prod(\text{hook lengths in } [\lambda])}.$$

We can define a non-degenerate symmetric bilinear form on M^λ as follows

$$(\{t_1\},\{t_2\}) = \begin{cases} 1 & \text{if } \{t_1\} = \{t_2\} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $char\mathbf{F} = p$ is a prime and λ is p-regular. Define $D^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda\perp})$.

Theorem 2.26 ([24], Theorem 11.5). *$\{D^\lambda \mid \lambda$ a p-regular partition$\}$ is a complete system of inequivalent irreducible $\mathbf{F}S_n$-modules. Moreover, D^λ with λ p-regular, is self-dual and absolutely irreducible, and every field is a splitting field for S_n.*

We record here some results on the decomposition of the Specht modules S^λ and the degrees of D^λ for some small partitions.

Lemma 2.27 ([24], Theorems 24.1, 24.15). *Let $\mathbf{F}$ be a field of characteristic p. Then*
(i) $S^{(n-1,1)} = D^{(n-1,1)} + \varepsilon_p(n)D^{(n)}$, *and* $dim D^{(n-1,1)} = n - 1 - \varepsilon_p(n)$.
(ii) if $p > 2$, *then* $S^{(n-2,1^2)} = D^{(n-2,1^2)} + \varepsilon_p(n)D^{(n-1,1)}$ *and*

$$dim D^{(n-2,1^2)} = \begin{cases} \frac{1}{2}(n-1)(n-2), & \text{if } p \nmid n \\ \frac{1}{2}(n-2)(n-3), & \text{if } p \mid n. \end{cases}$$

(iii) if $p > 2$, *then* $S^{(n-2,2)} = D^{(n-2,2)} + \varepsilon_p(n-2)D^{(n-1,1)} + \varepsilon_p(n-1)D^{(n)}$, *and*

$$dim D^{(n-2,2)} = \begin{cases} \frac{1}{2}n(n-3), & \text{if } n \not\equiv 1, 2 \ (mod\ p) \\ \frac{1}{2}(n^2-3n-2), & \text{if } n \equiv 1 \ (mod\ p) \\ \frac{1}{2}(n^2-5n+2), & \text{if } n \equiv 2 \ (mod\ p) \end{cases}$$

(iv) If $p = 2$, *then* $S^{(n-2,2)} = D^{(n-2,2)} + \varepsilon_p(n-2)D^{(n-1,1)} + \delta D^{(n)}$, *and*

$$dimD^{(n-2,2)} = \begin{cases} \frac{1}{2}(n-1)(n-4), & \textit{if } n \equiv 0 \textit{ (mod 4)} \\ \frac{1}{2}(n^2-3n-2), & \textit{if } n \equiv 1 \textit{ (mod 4)} \\ \frac{1}{2}(n^2-5n+2), & \textit{if } n \equiv 2 \textit{ (mod 4)} \\ \frac{1}{2}n(n-3), & \textit{if } n \equiv 3 \textit{ (mod 4)} \end{cases}$$

where

$$\delta = \begin{cases} 1, & \textit{if } n \equiv 1,2 \textit{ (mod 4)} \\ 0, & \textit{otherwise.} \end{cases}$$

For $k \geqslant 1$, denote by $R_n(k)$ the set of irreducible S_n-modules D such that $D \cong D^\lambda$ or $D \cong D^\lambda \otimes S^{(1^n)}$ for some p-regular partition $\lambda \vdash n$, with $\lambda_1 \geqslant n-k$.

Theorem 2.28 ([25], [39] Proposition 2.2). *Let* $n \geqslant 12$ *if* $p \neq 2$, *and* $n \geqslant 17$ *if* $p = 2$. *Then any irreducible* $\mathbf{F}S_n$*-module* D *either belongs to* $R_n(2)$ *or* $dimD > (n-2)(n-3)$.

We will need information about the restrictions to S_{n-1}, A_{n-1} of modular irreducible representations of symmetric groups S_n, and alternating groups A_n. The result will be useful in determine the maximality of members of $\mathcal{S}$, whose socles are alternating groups. For $i \in \{1, \dots, h\}$, we define $\lambda(i) = (\lambda_1^{a_1}, \lambda_2^{a_2}, \dots, \lambda_{i-1}^{a_i-1}, \lambda_i^{a_i-1}, \lambda_i - 1, \lambda_{i+1}^{a_i+1}, \dots, \lambda_h^{a_h})$, a partition of $n-1$. With above notations, we can state the branching rules for symmetric groups.

Theorem 2.29 ([31], Theorem 0.3). *Let* $\lambda \vdash n$ *be a* p*-regular partition. Then* $D^\lambda \downarrow_{S_{n-1}}$ *is irreducible if and only if* λ *is a JS-partition. If this is the case, then* $D^\lambda \downarrow_{S_{n-1}} = D^{\lambda(1)}$.

Next we consider the restriction to A_n of D^λ. In order to determine $D^\lambda \downarrow_{A_n}$, we need to find the fixed points under the *Mullineux map* $m(\lambda)$, originally defined in [40]. Following Ben Ford in [10], we will use the *p-modular Frobenius symbol* to find the fixed points of $m(\lambda)$. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \vdash n$ be a partition of n. The *rim* of the Young diagram $[\lambda]$

of λ is the collection of nodes which are either at the bottom of a column, at the right end of a row, or both. The *p-rim* of $[\lambda]$ is defined as follows: Beginning at the top right-hand corner of $[\lambda]$, the first p nodes of the rim are in the p-rim. Then skip to the next row, and take the next p nodes of the rim. Continue until we reach the end of the rim. The last p-segment may contain fewer than p nodes. Let h_1 be the number of nodes in the p-rim of λ, and let r_1 be the number of rows in λ. Delete the p-rim and repeat the process to get $h_1, r_1, \ldots, h_k, r_k$, where $h_{k+1} = r_{k+1} = 0$, but $h_k \neq 0 \neq r_k$. The *Mullineux symbol* is a $2 \times k$ matrix,

$$M(\lambda) = \begin{pmatrix} h_1 & h_2 & \cdots & h_k \\ r_1 & r_2 & \cdots & r_k \end{pmatrix}.$$

Now, the p-regular partition $m(\lambda)$ of n is defined via

$$M(m(\lambda)) = \begin{pmatrix} h_1 & h_2 & \cdots & h_k \\ s_1 & s_2 & \cdots & s_k \end{pmatrix},$$

where

$$\varepsilon_i = \begin{cases} 0, & \text{if } p \mid h_i \\ 1, & \text{if } p \nmid h_i \end{cases}$$

and $s_i = h_i - r_i + \varepsilon_i$. Note that the partition $m(\lambda)$ can be reconstructed from the Mullineux symbol $M(m(\lambda))$. The following notations will be useful if we just want to know whether a given partition is fixed under the Mulllineux map. The *p-modular Frobenius symbol* for λ, denoted by $Fr_p(\lambda)$, is a $3 \times k$ matrix

$$Fr_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k \end{pmatrix}$$

constructed as follows:

$$\begin{cases} a_i &= h_i - r_i \\ b_i &= r_i - \varepsilon_i \end{cases}$$

If λ has p-modular Frobenius symbol $Fr_p(\lambda)$ as constructed above then the Mullineux map m is defined by

$$Fr_p(m(\lambda)) = \begin{pmatrix} b_1 & b_2 & \cdots & b_k \\ a_1 & a_2 & \cdots & a_k \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k \end{pmatrix}$$

which means that we interchange the first two rows of $Fr_p(\lambda)$. Therefore, we see that λ is a fixed point of m if and only if the first two rows of $Fr_p(\lambda)$ are the same (see [10]). Denote by sgn_n, the sign character of S_n, which takes value 1 at even permutation and -1 at odd permutation. Now, if λ is a p-regular partition of n then $D^\lambda \bigotimes sgn_n = D^{m(\lambda)}$. This is the Mullineux conjecture, which was finally proved in [11]. Let λ be a 3-regular partition of n with $\lambda_1 \geqslant n-2$. Then λ is rarely a fixed point of Mullineux map.

Lemma 2.30 *Assume that $p = 3, n \geqslant 5$ and λ is a p-regular partition of n. Suppose that $\lambda_1 \geqslant n-2$. Then $m(\lambda) = \lambda$ if and only if $5 \leqslant n \leqslant 6$ and $\lambda = (n-2, 1^2)$.*

Proof. As $\lambda_1 \geqslant n-2$, the possibilities for λ_1 are $n, n-1$ or $n-2$. It follows that $\lambda = (n), (n-1, 1), (n-2, 2)$ or $(n-2, 1^2)$. If one of the first three cases holds then the result follows by Lemma 1.8 in [30]. Assume that $\lambda = (n-2, 1^2)$. We first compute the Mullineux symbol $M(\lambda)$ of λ. We have

$$M(\lambda) = \begin{pmatrix} 5 & 3 & \cdots & 3 & a \\ 3 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

where 3, and so 1, occurs t times with $t = \left[\frac{n-2}{3}\right] - 1$, and $0 \leqslant a = n - 2 - 3(t+1) \leqslant 2$. Hence

$$Fr_3(\lambda) = \begin{pmatrix} 2 & 2 & \cdots & 2 & a-1 \\ 2 & 1 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

If $t \geqslant 1$, or equivalently $n \geqslant 8$, then clearly, the first two rows of $Fr_3(\lambda)$ cannot be equal, so that $\lambda \neq m(\lambda)$. Thus $5 \leqslant n \leqslant 7$. Then

$$Fr_3(\lambda) = \begin{pmatrix} 2 & a-1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Observe that the first two rows of $Fr_3(\lambda)$ are equal if and only if $a = 0$ or $a = 1$. Since $t = 0$, $a = n - 2 - 3 = n - 5$, so that $n = 5$ or 6. ■

Theorem 2.31 ([10], Theorem 2.1). *Let* $\mathbf{F}$ *be a splitting field for* A_n *of characteristic* $p > 2$, *and* $\lambda \vdash n$ *be a* p*-regular partition. Then*
(i) If $\lambda \neq m(\lambda)$, *then* $D^\lambda \downarrow_{A_n} = D^{m(\lambda)} \downarrow_{A_n}$ *is irreducible.*
(ii) If $\lambda = m(\lambda)$, *then* $D^\lambda \downarrow_{A_n}$ *is a sum of two irreducible, non-equivalent representations of* A_n, *say* D_+^λ *and* D_-^λ, *interchanging by an odd permutation.*
Finally

$$\{D^\lambda \downarrow_{A_n} | \lambda \neq m(\lambda)\} \cup \{D_+^\lambda, D_-^\lambda \mid \lambda = m(\lambda)\}$$

is a complete system of inequivalent irreducible $\mathbf{F}A_n$*-modules.*

We next prove a gap result between the minimal module and the second minimal module for Alternating groups in characteristic 3.

Lemma 2.32 *Let* $\mathbf{F}$ *be a splitting field for* A_n *of characteristic* $p = 3$. *Suppose that* $n \geqslant 12$ *and* V *is an irreducible* $\mathbf{F}A_n$*-module with* $dimV > n$. *Then* $dimV \geqslant \frac{1}{2}(n^2 - 5n + 2)$.

Proof. It follows from Theorem 2.31 that either $V = D^{\lambda}\downarrow_{A_n}$ with $m(\lambda) \neq \lambda$ or $V = D^{\lambda}_{\pm}$, where $m(\lambda) = \lambda$. Now, let $U = D^{\lambda}$ for partition λ obtained above. By Theorem 2.28, either $\dim U > (n-2)(n-3)$, or $U \in R_n(2)$. Observe that $dimV = dimU$ if $m(\lambda) \neq \lambda$ and if $m(\lambda) = \lambda$ then $dimV = \frac{1}{2}dimU$. Thus, if $dimU > (n-2)(n-3)$, then clearly, $\dim V \geqslant \frac{1}{2}dimU > \frac{1}{2}(n-2)(n-3) > \frac{1}{2}(n^2-5n+2)$. Therefore, we can assume that $U \in R_n(2)$. Thus there exists a 3-regular partition μ with $\mu(1) \geqslant n-2$ such that $\lambda = \mu$ or $\lambda = m(\mu)$. As $n > 10$ and $dimV > n$, it follows from Proposition 2.23 that μ is not (n) nor $(n-1,1)$, and so $\mu = (n-2,2)$, or $(n-2,1^2)$. By Lemma 2.30, μ is not fixed under the Mullineux map. Also, as Mullineux map is an involutory map, $m(m(\mu)) = \mu \neq m(\mu)$. We conclude that D^{μ} and $D^{m(\mu)}$ are irreducible upon restricted to A_n. Since $dimD^{m(\mu)} = dimD^{\mu} \otimes sgn_n = dimD^{\mu}$, we have $dimV = dimD^{\lambda} = dimD^{\mu}$. Finally, the result follows from Lemma 2.27. ∎

Continue the hypotheses in Theorem 2.31, let $\lambda \vdash n$ be a p-regular partition with p odd. Define $D^{\lambda^{\circ}} = D^{\lambda}\downarrow_{A_n}$ if $\lambda \neq m(\lambda)$ and $D^{\lambda^{\circ}} \in \{D^{\lambda}_{+}, D^{\lambda}_{-}\}$, if λ is a fixed point of the Mullineux map. The branching rule for A_n is given in the following theorem.

Theorem 2.33 ([4], Theorem 5.10). *Let $n \geqslant 2, p$ an odd prime, and λ be a p-regular partition of n. Then the following are equivalent:*

(i) $D^{\lambda^{\circ}}\downarrow_{A_{n-1}}$ *is irreducible;*

(ii) One of the following holds:

 (a) $D^{\lambda}\downarrow_{S_{n-1}}$ *is irreducible;*

 (b) $D^{\lambda}\downarrow_{S_{n-1}} = D^{\lambda(1)} \bigoplus D^{m(\lambda(1))}$ *and* $\lambda = m(\lambda)$ *but* $\lambda(1) \neq m(\lambda(1))$.

2.5 Representations of Finite groups of Lie type in cross characteristic

The irreducible cross characteristic representations of $L_2(q)$ are given in Table 2.7. These are taken from [19]. For notations in this table, the columns $'d'$ are the degrees of the representations, the columns $'\ell'$ are the characteristic of the fields of representations, and the 'field' columns give the irrationalities of the Brauer characters.

Landazuri and Seitz [32] and Seitz and Zalesskii [43] obtained the lower bound for the irreducible cross characteristic representations of finite simple Chevalley groups. This result has been improved by many other authors including Guralnick, Magaard, Saxl and Tiep [16], Guralnick and Tiep [15], [17], and Hoffman [20]. Table 2.6 is taken from [20].

Theorem 2.34 *Assume that L is a finite simple Chevalley group in characteristic p. Then $R_{p'}(L) \geqslant e(L)$, where $e(L)$ is as in Table 2.6.*

2.6 Representations of Finite groups of Lie type in defining characteristic

Simple algebraic groups

Let k be an algebraically closed field of characteristic p. A *simple algebraic group* over k is a linear algebraic group over k which has no proper, closed, connected normal subgroups. The simple algebraic groups over k were classified and they are the groups of types $A_\ell(k), B_\ell(k), C_\ell(k), D_\ell(k), E_\ell(k)(l = 6, 7, 8), F_4(k)$ and $G_2(k)$. For each type, there is a unique *simply connected* group whose center Z is as large as possible, and an *adjoint* group with trivial center. Let G be a simple algebraic group over k, and let $q = p^f$. Regard G as a subgroup of $GL_n(k)$ for some n. Let σ_q be a map from G to G sending (a_{ij})

Table 2.6: Bounds for Cross Characteristic Representations of Chevalley Groups.

L	$e(L)$	exceptions
$PSL_2(q)$	$(q-1)/(2,q-1)$	$e(PSL_2(4)) = 2$,
		$e(PSL_2(9)) = 3$
$PSL_n(q), n \geqslant 3$	$(q^n - q)/(q-1) - 1$	$e(PSL_3(2)) = 2$,
		$e(PSL_3(4)) = 4$,
		$e(PSL_4(2)) = 7$,
		$e(PSL_4(3)) = 26$
$PSp_{2n}(q), n \geqslant 2$	$(q^n - 1)/2, q$ odd	
	$(q^n - 1)(q^n - q)/(2(q+1)), q$ even	$e(PSp_4(2)') = 2$
$PSU_n(q), n \geqslant 3$	$(q^n - q)/(q+1), n$ odd	
	$(q^n - 1)/(q+1), n$ even	$e(PSU_4(2)) = 4$,
		$e(PSU_4(3)) = 6$
$P\Omega^+_{2n}(q), n \geqslant 4$	$(q^n - 1)(q^{n-1} + q)/(q^2 - 1) - 2, q > 3$	
	$(q^n - 1)(q^{n-1} - 1)/(q^2 - 1), q \leqslant 3$	$e(P\Omega^+_8(2)) = 8$
$P\Omega^-_{2n}(q), n \geqslant 4$	$(q^n + 1)(q^{n-1} - q)/(q^2 - 1) - 1$	$e(P\Omega^-_8(2)) \geqslant 32$,
		$e(P\Omega^-_8(4)) \geqslant 1026$,
		$e(P\Omega^-_{10}(2)) \geqslant 151$,
		$e(P\Omega^-_{10}(3)) \geqslant 2376$
$P\Omega_{2n+1}(q), n \geqslant 3$	$(q^{2n} - 1)/(q^2 - 1) - 2, q > 3, q$ odd	
	$(q^n - 1)(q^n - q)/(q^2 - 1), q = 3$	$e(P\Omega_7(3)) = 27$
$E_6(q)$	$q\Phi_8(q)\Phi_9(q) - 2$	
$E_7(q)$	$q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q) - 3$	
$E_8(q)$	$q\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q) - 4$	
$F_4(q)$	$q^4(q^6 - 1), q$ odd	
	$q^7(q^3 - 1)(q-1)/2, q$ even	$e(F_4(2)) = 52$
${}^2E_6(q)$	$q^9(q^2 - 1)$	
$G_2(q)$	$q(q^2 - 1)$	$e(G_2(3)) = 14$,
		$e(G_2(4)) = 12$
${}^3D_4(q)$	$q^3(q^2 - 1)$	
${}^2F_4(q)$	$q^4\sqrt{q/2}(q-1)$	
$Sz(q)$	$\sqrt{q/2}(q-1)$	$e(Sz(8)) = 8$
${}^2G_2(q)$	$q(q-1)$	

Table 2.7: Irreducible representations of $L_2(q)$.

(a) $L_2(q), q \equiv 1(\text{mod } 4), \ell \nmid q$

degree	G	ℓ	field	ind
$\frac{q-1}{2}$	$L_2(q)$	2	$\sqrt{q}$	$-$
$\frac{q-1}{2}$	$2.L_2(q)$	$\ell \neq 2$	$\sqrt{q}$	$-$
$\frac{q+1}{2}$	$L_2(q)$	$\ell \neq 2$	$\sqrt{q}$	$+$
$q-1$	$L_2(q)$		$\zeta_{q+1}^{j} + \zeta_{q+1}^{-j}$	$+$
$q-1$	$2.L_2(q)$	$((q+1)/2)_{\ell'} \neq 1$	$\zeta_{q+1}^{j} + \zeta_{q+1}^{-j}$	$-$
q	$L_2(q)$	$\ell \nmid (q+1)$		$+$
$q+1$	$L_2(q)$	$((q-1)/4)_{\ell'} \neq 1$	$\zeta_{q-1}^{2j} + \zeta_{q-1}^{-2j}$	$+$
$q+1$	$2.L_2(q)$	$\ell \neq 2$	$\zeta_{q-1}^{j} + \zeta_{q-1}^{-j}$	$-$

(b) $L_2(q), q \equiv 3(\text{ mod } 4), \ell \nmid q$

degree	G	ℓ	field	ind
$\frac{q-1}{2}$	$L_2(q)$		$\sqrt{-q}$	$\circ$
$\frac{q+1}{2}$	$2.L_2(q)$	$\ell \neq 2$	$\sqrt{-q}$	$\circ$
$q-1$	$L_2(q)$		$\zeta_{q+1}^{2j} + \zeta_{q+1}^{-2j}$	$+$
$q-1$	$2.L_2(q)$	$\ell \neq 2$	$\zeta_{q+1}^{j} + \zeta_{q+1}^{-j}$	$-$
q	$L_2(q)$	$\ell \nmid (q+1)$		$+$
$q+1$	$L_2(q)$	$((q-1)/2)_{\ell'} \neq 1$	$\zeta_{q-1}^{j} + \zeta_{q-1}^{-j}$	$+$
$q+1$	$2.L_2(q)$	$((q-1)/2)_{\ell'} \neq 1, \ell \neq 2$	$\zeta_{q-1}^{j} + \zeta_{q-1}^{-j}$	$-$

(c) $L_2(q), q \equiv 0(\text{ mod } 2), \ell \nmid q$

degree	G	ℓ	field	ind
$q-1$	$L_2(q)$		$\zeta_{q+1}^{j} + \zeta_{q+1}^{-j}$	$+$
q	$L_2(q)$	$\ell \nmid (q+1)$		$+$
$q+1$	$L_2(q)$	$(q-1)_{\ell'} \neq 1$	$\zeta_{q-1}^{j} + \zeta_{q-1}^{-j}$	$+$

to (a_{ij}^q). Then σ_q is called a *standard Frobenius map.* In general, a map $\sigma : G \to G$ is said to be a *Frobenius map* if some power of σ is a standard Frobenius map. Steinberg ([46], Corollary 10.13) showed that if $\sigma : G \to G$ is surjective then either σ is an automorphism or G_σ is finite. Now let σ be a Frobenius map and let G_σ be the fixed point group of σ in G. Clearly, σ is a surjective map but it is not an automorphism. Hence G_σ is finite, and the finite groups $O^{p'}(G_\sigma)$ are called the *finite groups of Lie type in characteristic p.* A *torus* of the simple algebraic group G over k is a subgroup which is isomorphic to a direct product of copies of k^*. Every torus lies in a maximal torus and all maximal tori are conjugate in G. Each maximal torus T is isomorphic to ${k^*}^\ell$, where ℓ is the Lie rank of G, and $C_G(T) = T$. The group $W = N_G(T)/T$ is finite and is called the *Weyl group* of G. A *Borel subgroup* of G is a maximal closed, connected, solvable subgroup of G. It is shown that all Borel subgroups are conjugate in G, and the maximal tori of G are those of the Borel subgroups of G. Fix a Borel subgroup B and a maximal torus T of G. Let U be the *unipotent radical* of B, which consists of all unipotent elements of the largest connected normal solvable subgroup of B. Then B is self-normalizing, $B = N_G(U) = U : T$, and there exists a unique Borel subgroup of G, denoted by B^-, such that $B \cap B^- = T$, called *opposite* B (relative to T). (cf. [22], $21.3, 23.1, 26.2$).

Modules and weights

Let L be a simply connected group of Lie type over $\mathbf{F}_q$, where $q = p^f$. Thus $L = O^{p'}(G_\sigma)$, where G is a simply connected, simple algebraic group over k, and σ is a suitable Frobenius map. Fix a maximal torus T and a Borel subgroup B of G. Let $X = X(T)$ be the *character group of* T, the set of algebraic group homomorphisms from T to k^*. If M is any irreducible *rational kG-module*, that is, the corresponding map $G \to GL(M)$ is an algebraic group homomorphism, then $M = \bigoplus_{\mu \in X} M_\mu$, where for $\mu \in X$, we have $M_\mu = \{v \in M \mid vt = \mu(t)v \text{ for all } t \in T\}$. If $M_\mu \neq 0$, then μ is said to be a *weight* of M, and M_μ the *μ-weight space* of M. The Weyl group $W = N_G(T)/T$ acts on X and induces a group of permutations on the weights of M. As X is abelian, it is a $\mathbb{Z}$-module. Thus we may form the tensor product $E = \mathbb{R} \bigotimes_{\mathbb{Z}} X$, and the action of W on X yields an action of W on E. Choose a positive definite, bilinear, symmetric W-invariant $\mathbb{R}$-form $(.,.)$ on E.

Assume that $\mathcal{L}$ is the *adjoint module* for G, that is, $\mathcal{L}$ is the Lie algebra of G over k, with the natural G-action. The set Φ of weights of $\mathcal{L}$ is called the set of *roots* of G. Select a system Φ^+ of *positive* roots from Φ, with corresponding set $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ of *fundamental roots,* giving the Dynkin diagram as in Figure 2.6. Define a partial order on X by writing $\mu \leqslant \lambda$ if and only if $\lambda - \mu$ is a sum of positive roots. For $\alpha \in \Phi, \alpha^* = \dfrac{2\alpha}{(\alpha, \alpha)}$ is the *co-root* corresponding to α. As L is simply connected, there is a basis $\{\lambda_1 \ldots, \lambda_\ell\}$ of E which is dual to $\{\alpha_1^*, \ldots, \alpha_\ell^*\}$, that is, with $(\lambda_i, \alpha_j^*) = \delta_{ij}$. The λ_i form a $\mathbb{Z}$-basis for X, and they are called the *fundamental dominant weights.* Define $X^+ = \{\sum_{i=1}^{\ell} c_i\lambda_i \mid c_i \in \mathbb{Z}, c_i \geqslant 0\}$ and $X_q = \{\sum_{i=1}^{\ell} c_i\lambda_i \mid c_i \in \mathbb{Z}, 0 \leqslant c_i \leqslant q-1\}$. Let L be a group of type ${}^2B_2, {}^2G_2$ or ${}^2F_4$, over $\mathbf{F}_{p^{2a+1}}$, where $p = 2$ or 3. For a root α, set

$$q(\alpha) = \begin{cases} p^a, & \text{if } \alpha \text{ is a long root,} \\ p^{a+1}, & \text{if } \alpha \text{ is a short root.} \end{cases}$$

We define $X'_q = \{\sum_{i=1}^{\ell} c_i\lambda_i \mid c_i \in \mathbb{Z}, 0 \leqslant c_i \leqslant q(\alpha_i) - 1\}$. Each member of X^+ is called a *dominant weight.*

The irreducible rational G-modules are characterized by their highest weights.

Theorem 2.35 ([22], 31.1). *Let M be an irreducible (rational) kG-module.*
(a) There is a unique B-stable 1*-dimentional subspace, spanned by a vector $v^+ \in M$,* (a maximal vector), *with dominant weight λ, whose multiplicity is* 1, *and* $dim(M_\lambda) = 1$. *For all other weights μ of M, $\mu = \lambda - \sum_{i=1}^{\ell} c_i\lambda_i \leqslant \lambda$, where* $c_i \in \mathbb{Z}, c_i \geqslant 0$. *These weights are permuted by the Weyl group W, and W-conjugate weights have the same multiplicity.*
(b) Irreducible kG-modules are determined up to isomorphism by their highest weights.

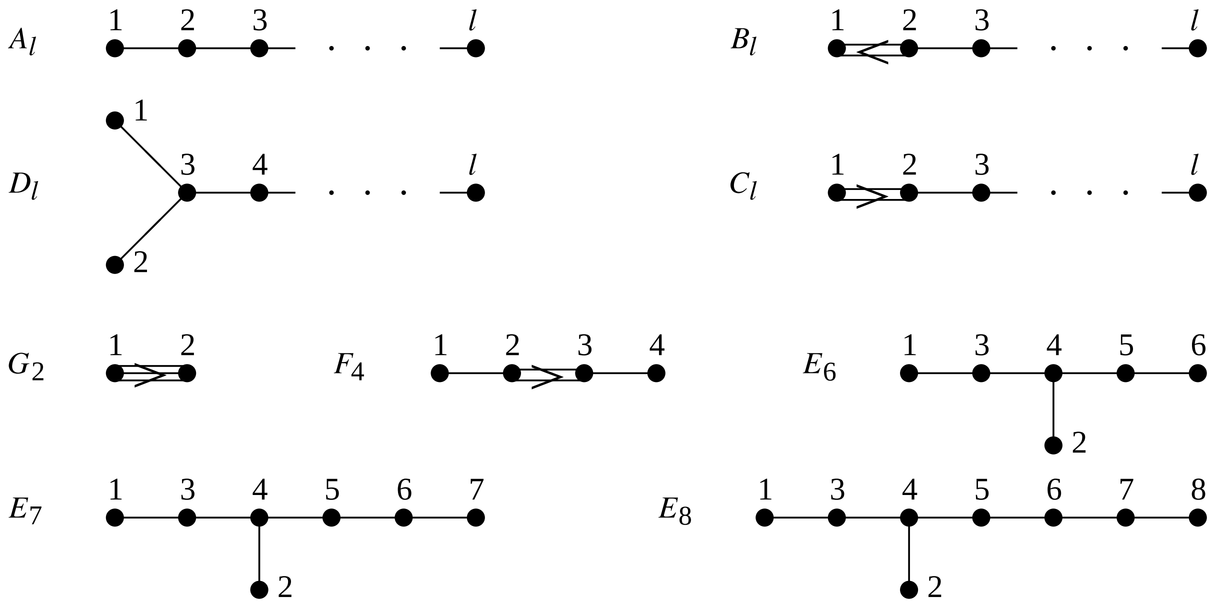

Figure 2.1: Dynkin diagram.

(c) *For each dominant weight* $\lambda \in X$*, there exists an irreducible* kG*-module* $L(\lambda)$ *of highest weight* λ*.*

The unique dominant weight λ in this theorem is called the *highest weight* of M. The following theorem of Steinberg shows us how to get kL-modules from kG-modules.

Theorem 2.36 ([46],§13). *Let* $\Lambda = X'_q$ *if* L *is of type* ${}^2B_2, {}^2G_2$ *or* ${}^2F_4$*, and* $\Lambda = X_q$ *otherwise. Then for* $\lambda \in \Lambda$*, the modules* $L(\lambda)$ *remain irreducible and inequivalent upon restriction to* L*, and exhaust the irreducible* kL*-modules.*

Let M be a kL-module affording a representation $\rho : L \to GL_n(k)$, and ν be the automorphism of $GL_n(k)$ induced by the action of the field automorphism $t \mapsto t^p$ on matrix entries. For $r \geqslant 1$, let $M^{(r)}$ be the space M with L-action given by the representation $\rho\nu^r$. A dominant weight is called *p-restricted* if it lies in X_p. Steinberg's twisted tensor product theorem shows how to construct all highest weight modules $L(\lambda)$ from those of p-restricted highest weights.

Theorem 2.37 ([44], Steinberg's Twisted Tensor Product Theorem). *Let* $\lambda = \mu_0 + p\mu_1 + \cdots + p^{e-1}\mu_{e-1} \in X_q$, *where* μ_i *are p-restricted for all* i, *and* $q = p^e$. *Then* $L(\lambda) = L(\mu_0) \otimes L(\mu_1)^{(1)} \otimes \cdots \otimes L(\mu_{e-1})^{(e-1)}$.

Let w_0 be the *longest element* of the Weyl group W of G, w_0 can be characterized as the unique element of W such that $w_0(\Phi^+) = \Phi^-$, where the root system Φ is the disjoint union of Φ^+ and Φ^-. Elements of Φ^- are called *negative roots*. Let L be of type A_ℓ, D_ℓ, E_6 or D_4 over $\mathbf{F}_q$, L possesses a graph automorphism $\tau_\circ$ of order $2, 2, 2$, or 3, respectively, which induces a symmetry τ on the Dynkin diagram.

$$w_0 = \begin{cases} -1 & \text{for types } B_\ell, C_\ell, D_\ell \ (\ell \text{ even}),\ G_2, F_4, E_7, E_8, \\ -\tau & \text{for types } A_\ell, D_\ell \ (\ell \text{ odd}),\ E_6. \end{cases}$$

Proposition 2.38 ([42], 1.8). $L(\lambda)^* \cong L(-w_0\lambda)$.

Proposition 2.39 ([42], 1.11). *Let* $\lambda = q_1\mu_1 + \cdots + q_k\mu_k$, *where* $\mu_1, \ldots, \mu_k$ *are p-restricted dominant weights and* $q_1, \ldots, q_k$ *are pairwise distinct powers of* p. *Then* G *leaves invariant a non-degenerate bilinear form on* $L(\lambda)$ *if and only if* G *leaves such a form invariant on each of* $L(\mu_1), \ldots, L(\mu_k)$.

Proposition 2.40 ([45]) *Let* L *be a group of Lie type over* $\mathbf{F}_q$.

(i) If L *is untwested or of type* ${}^2B_2, {}^2G_2$ *or* ${}^2F_4$, *then* $\mathbf{F}_q$ *is a splitting field for* L.

(ii) If L *is of type* ${}^2A_\ell, {}^2D_\ell$ *or* ${}^2E_6$, *then* $\mathbf{F}_{q^2}$ *is a splitting field for* L.

(iii) If L *is of type* ${}^3D_4$, *then* $\mathbf{F}_{q^3}$ *is a splitting field.*

Proposition 2.41 ([29], Theorem 5.4.5). *Let* L *be simply connected of Lie type over* $\mathbf{F}_{p^e}$, *and suppose that* V *is an absolutely irreducible* $\mathbf{F}_{p^f}L$*-module which is realized over no proper subfield of* $\mathbf{F}_{p^f}$.

(i) If L is untwisted, then $f \mid e$ and there is an irreducible $kL-$module M such that

$$V \otimes k \cong M \otimes M^{(f)} \otimes \cdots \otimes M^{(e-f)}.$$

In particular, $dim(V) = dim(M)^{e/f}$.

(ii) If L is of type ${}^2A_\ell$, ${}^2D_\ell$ or ${}^2E_6$, then one of the following occurs.

(a) $f \mid e$, $V \cong V^{\tau_\circ}$, there is an irreducible kL-module M such that $M \cong M^{\tau_\circ}$ and

$$V \otimes k \cong M \otimes M^{(f)} \otimes \cdots \otimes M^{(e-f)}.$$

In particular, $dim(V) = dim(M)^{e/f}$.

(b) $f \mid 2e$ but f does not divide e, $V \ncong V^{\tau_\circ}$, $V^{(f/2)} \cong V^{\tau_\circ}$, there is an irreducible kL-module M such that $M \ncong M^{\tau_\circ}$ and

$$V \otimes k \cong M \otimes (M^{\tau_\circ})^{(f/2)} \otimes M^{(f)} \otimes (M^{\tau_\circ})^{(3f/2)} \otimes \cdots \otimes (M^{\tau_\circ})^{(e-f)} \otimes M^{(e-(e/f))}.$$

In particular, $dim(V) = dim(M)^{2e/f}$.

(iv) If L has type ${}^3D_4$, then either $f \mid e$ and (ii)(a) holds or $f \mid 3e$, but f does not divide e, and $V \otimes k$ is a tensor product of $3e/f$ modules $M \otimes (M^{\tau_\circ})^{(f/2)} \otimes M^{(f)} \otimes (M^{\tau_\circ})^{(3f/2)} \otimes \dots$ and $dimV = (dimM)^{(3e/f)}$.

(v) If L is of type ${}^2B_2, {}^2G_2$ or ${}^2F_4$ over $\mathbf{F}_q$, where $q = p^{2a+1}$ and p is $2, 3$ or 2, respectively, then $f \mid 2a+1$ and $dimV \geqslant R_p(L)^{(2a+1)/f}$, where $R_p(L)$ is $4, 7$ and 26 for types ${}^2B_2, {}^2G_2$ and ${}^2F_4$, respectively.

Chapter 3

Rank 3 permutation characters and maximal subgroups

3.1 Higman rank 3 parameters and the equation

Let G be a finite group acting transitively on a non-empty finite set $\mathfrak{E}$. Let P be a stabilizer of a point $x \in \mathfrak{E}$ in G. Recall that the action is primitive if and only if P is maximal in G. Also the rank of G is the number of orbits of P on $\mathfrak{E}$. Now suppose that G is of even order and acts primitively rank 3 on $\mathfrak{E}$. So P has exactly three orbits on $\mathfrak{E}$, namely, $\{x\}$, $\Delta(x)$ and $\mathit{\Gamma}(x)$. Define $k = |\Delta(x)|$, $l = |\mathit{\Gamma}(x)|$ and

$$|\Delta(x) \cap \Delta(y)| = \begin{cases} \mu & \text{if } y \in \mathit{\Gamma}(x) \\ \lambda & \text{if } y \in \Delta(x) \end{cases}$$

$$|\mathit{\Gamma}(x) \cap \mathit{\Gamma}(y)| = \begin{cases} \lambda_1 & \text{if } y \in \mathit{\Gamma}(x) \\ \mu_1 & \text{if } y \in \Delta(x). \end{cases}$$

Suppose that $k \leqslant l$.

Lemma 3.1 ([18], Lemma 5 and 7). *Let G act primitively rank 3 on $\mathfrak{E}$. Then*

(i) $|\mathfrak{E}| = k + l + 1$;

(ii) $\mu l = k(k-1-\lambda)$;

(iii) $D = (\lambda-\mu)^2 + 4(k-\mu)$ *is a square;*

(iv) $\lambda_1 = l - k + \mu - 1$;

(v) $\mu_1 = l - k + \lambda + 1$.

Let V be the permutation module for G on $\mathfrak{E}$ over $\mathbb{C}$, hence $\mathfrak{E}$ is a basis for V. Further Δ and Γ can be viewed as linear maps on V, via the corresponding $x \mapsto \sum_{y\in\Delta(x)} y$ and $x \mapsto \sum_{y\in\Gamma(x)} y$, for $x \in \mathfrak{E}$, and extend linearly. We have $\Sigma_{y\in\mathfrak{E}} y$ is an eigenvector for Δ and Γ, with eigenvalues k and l, respectively. Other eigenvalues for Δ and Γ are as follows:

Lemma 3.2 ([18], Lemma 6). *The eigenvalues for Δ are:*

$$s = \frac{\lambda-\mu+\sqrt{D}}{2} \quad \textit{and} \quad t = \frac{\lambda-\mu-\sqrt{D}}{2}.$$

Lemma 3.3 ([2], 1.4.3). *For $r \in \{s,t\}$, the eigenvalues for Γ are $-(r+1)$.*

Let V_r, $r = s$ or $r = t$, be the irreducible modules for G on V which is the eigenspace for Δ with eigenvalue r. Let $f_r = dim(V_r)$.

Lemma 3.4 ([18]). $f_s = \dfrac{k+t(k+l)}{t-s}$ *and* $f_t = \dfrac{k+s(k+l)}{s-t}$.

Let M be any subgroup of G. Fix $x \in \mathfrak{E}$, and let $P = G_x$. By identifying $\mathfrak{E}$ with G/P, x corresponds to the coset P in G/P. Consider the action of G on the right cosets G/M and form the permutation module V_M. Denote by y the coset M as a point in G/M. Define

$$d = d_x = |xM \cap \Delta(x)| \text{ and } \; c = c_x = |xM \cap \Gamma(x)|.$$

As G acts transitively on $\mathfrak{E}$ and G/M, $xG = \mathfrak{E}$ and $G/M = yG$. Define $\alpha : V \to V_M$ by

$$\alpha(xg) = \Sigma_{p\in P}\, ypg \text{ for } g \in G$$

and $\beta : V_M \to V$ by

$$\beta(yg) = \Sigma_{m \in M} xmg \text{ for } \ g \in G.$$

Then α and β are $\mathbb{C}G$ maps and $\theta = \frac{1}{|P||P \cap M|}.\beta \circ \alpha \in End_{\mathbb{C}G}(V)$. The map θ can be written in terms of linear maps Δ and $\mathit{\Gamma}$ as follows:

Lemma 3.5 ([2], (2.1)). $\theta = I + \dfrac{d\Delta}{k} + \dfrac{c\mathit{\Gamma}}{l}$

Proof. For any $g \in G$, we have

$$\theta(xg) = \frac{1}{|P||P \cap M|}.\beta(\alpha(xg)) = \frac{1}{|P||P \cap M|}.\sum_{p \in P} \beta(ypg) = \frac{1}{|P||P \cap M|}.\sum_{p \in P, m \in M} xmpg.$$

Let $\mathcal{O}$ be one of the three orbits of P on $\mathfrak{E}$ and $v_{\mathcal{O}} = \sum_{u \in \mathcal{O}} u \in V$. As P acts on $\mathcal{O}$, $xmp \in \mathcal{O}$ if and only if $xm \in \mathcal{O}$, in which case since P is transitive on $\mathcal{O}$,

$$\sum_{p \in P} xmp = \frac{|P|}{|\mathcal{O}|} v_{\mathcal{O}}.$$

Moreover there are $|M_x| = |P \cap M|$ elements in M fixing x and $|P \cap M|d, |P \cap M|c$ elements $m \in M$ with $xm \in \mathcal{O}$ for $\mathcal{O} = \Delta, \mathit{\Gamma}$, respectively. Therefore

$$\theta(xg) = (v_x + \frac{dv_{\Delta}}{k} + \frac{cv_{\mathit{\Gamma}}}{l})(x)g = (I + \frac{d\Delta}{k} + \frac{c\mathit{\Gamma}}{l})(xg).$$

This proves the lemma. ■

For $r = s, t$, let π_r be the projection of V on V_r.

Lemma 3.6 ([2], (2.2) and (2.3)). *Let $r = s$ or t, then*

(i) $\theta \circ \pi_r = 0$ if and only if $V_r \leqslant ker(\theta)$;

(ii) If $\theta \circ \pi_r \neq 0$ then $\alpha : V_r \to V_M$ is an injective $\mathbb{C}G$ map;

(iii) For $r = s, t : \theta \circ \pi_r = 0$ if and only if

$$1 + \frac{dr}{k} = \frac{(r+1)c}{l}. \tag{3.1}$$

Proof. (1) is clear as $\pi_r(V) = V_r$. By Lemma 3.4, V_s is not $\mathbb{C}G$-isomorphic to V_t, so that $\theta : V_r \to V_r$. As V_r is irreducible $\mathbb{C}G$-module, θ is an isomorphism if θ is non-zero. Thus (2) follows from (1). For (3), let $r = s, t$ and $v \in V$. Let $\Sigma = \Delta$ or $\mathit{\Gamma}$. Then

$$(\Sigma \circ \pi_r)(v) = e(\Sigma, r)\pi_r(v),$$

where $e(\Sigma, r)$ is the eigenvalue of Σ on V_r. Thus $\Sigma \circ \pi_r = e(\Sigma, r)\pi_r$. From definition, $e(r, \Delta) = r$ and by Lemma 3.3, $e(r, \mathit{\Gamma}) = -(r+1)$. Therefore by Lemma 3.5, we obtain:

$$\theta \circ \pi_r = (I + \frac{d\Delta}{k} + \frac{c\mathit{\Gamma}}{l}) \circ \pi_r = (1 + \frac{dr}{k} - \frac{(r+1)c}{l})\pi_r.$$

Thus $\theta \circ \pi_r = 0$ if and only if

$$1 + \frac{dr}{k} - \frac{(r+1)c}{l} = 0.$$

This finishes the proof. ■

As G is a primitive rank 3 group on $\mathfrak{E}$ with P the stabilizer of a point x in $\mathfrak{E}$, the permutation character 1_P^G has a decomposition $1_P^G = 1 + \chi_s + \chi_t$, where 1 is the trivial character, and χ_s, χ_t are irreducible characters of G, afforded by the irreducible modules V_s, V_t, with degrees f_s, f_t, respectively. We say that $1_P^G \leqslant 1_M^G$ if $1_M^G - 1_P^G$ is a character of G. This is equivalent to $(\chi_r, 1_M^G) > 0$ for any $r \in \{s, t\}$. By Lemma 3.6, for $r \in \{s, t\}$, $(\chi_r, 1_M^G) = 0$ if and only if equation (3.1) holds for any $x \in \mathfrak{E}$. Note that the parameters c, d depend on x, or equivalently, on the conjugate of P. When we pick different conjugate of P, parameters c and d change. Thus for $r \in \{s,t\}$, if equation (3.1) does not hold for some point $x \in \mathfrak{E}$ or some conjugate of P, then by Lemma 3.6, there is an injective $\mathbb{C}G$ map from V_r to V_M and hence $1_P^G \leqslant 1_M^G$. Otherwise we need to change to different conjugate of P or different point in $\mathfrak{E}$. See Proposition 3.20 for such an example.

The following result will be used frequently to show that there is no containment if M has at most 2 orbits on $\mathfrak{E}$.

Corollary 3.7 *Let G be a primitive rank 3 group acting on a finite set $\mathfrak{E}$. Let P be the stabilizer of a point in $\mathfrak{E}$, and M be any subgroup of G. If M has at most two orbits on $\mathfrak{E}$, then $1_P^G \nleqslant 1_M^G$.*

Proof. By way of contradiction, suppose that $1_P^G \leqslant 1_M^G$. Then $\phi = 1_M^G - 1_P^G$ is a character of G. Thus $(1_P^G, 1_M^G) = (1_P^G, \phi + 1_P^G) = (1_P^G, 1_P^G) + (1_P^G, \phi) = 3 + (1_P^G, \phi) \geqslant 3$, since $\phi, 1_P^G$ are characters of G. By Lemma 2.21, $(1_P^G, 1_M^G)$ is the number of orbits of M on G/P. Now, by identifying $\mathfrak{E}$ with G/P, M has $(1_P^G, 1_M^G) \geqslant 3$ orbits on $\mathfrak{E}$. A contradiction. ∎

3.2 An example

Let $L = \Omega_5(3)$. By Theorem 2.16, we have isomorphisms: $\Omega_5(3) \cong PSp_4(3) \cong PSU_4(2)$. Information on the maximal subgroups of L, extracted from [9], is given in Table 3.1. Let x be a minus point and P the stabilizer in L of x. From Table 3.1, we have $P \cong S_6$ and the permutation character of P in L can be decomposed as $1_P^L = 1a + 15b + 20a$. It follows that L has exactly 3 orbits on the cosets L/P. Since P is maximal in L, L acts as a primitive rank 3 permutation group on L/P. Let M be any maximal subgroup of L which is not conjugate to P. For example, take M to be the stabilizer of an isotropic point. Then $M \cong 3^3 : S_4$, and $1_M^L = 1a + 15b + 24a$. We have $1_M^L - 1_P^L = 24a - 20a$, which is not a character of L. Thus $1_P^L \nleqslant 1_M^L$. In fact, for any maximal subgroup M of L which is not conjugate to P, $1_P^L \nleqslant 1_M^L$. Note that L is also a primitive rank 3 permutation

Table 3.1: Maximal subgroups of $\Omega_5(3)$

Order	Index	Structure	Character	Specification
960	27	$2^4 : A_5$	$1a + 6a + 20a$	base
720	36	S_6	$1a + 15b + 20a$	minus point
648	40	$3_+^{1+2} : 2A_4$	$1a + 15a + 24a$	isotropic line
648	40	$3^3 : S_4$	$1a + 15b + 24a$	isotropic point
576	45	$2 \cdot (A_4 \times A_4).2$	$1a + 20a + 24a$	plus point

group on the cosets L/M for any maximal subgroup M of L. In general, we expect more containments.

3.3 Main Hypothesis and Notations

From now on, we assume the following set up. Let L be one of the following quasi-simple groups $\Omega_{2m+1}(3)$, or $\Omega^{\varepsilon}_{2m}(3)$ with $m \geqslant 2$ or $m \geqslant 3$, respectively. Let V be the natural module for L over $\mathbf{F} = \mathbf{F}_3$ with a non-degenerate quadratic form Q. Denote by $\mathfrak{E}(V)$ an L-orbit of non-singular points in V. Let G be a group such that G acts as a primitive rank 3 group on $\mathfrak{E}(V)$, and $L \trianglelefteq G, G/Z(L) \leqslant Aut(L/Z(L))$. We say that G is a *nearly simple primitive rank* 3 *group of type* L. Observe that $L \trianglelefteq G \leqslant I(V)$, where $I(V)$ is the full isometric group of V. Denote by P the stabilizer of a point in $\mathfrak{E}(V)$. The letter M will be reserved for the maximal subgroup of G. If $M \in \mathcal{C}(G)$ then for any subgroup X satisfying $\Omega(V) \leqslant X \leqslant \Xi(V)$, the X-associate of M is denoted by M_X. If $M \in \mathcal{S}(G)$ then we denote the socle of M by S. Then S is a non-abelian finite simple group and the full covering group $\widehat{S}$ of S acts absolutely irreducibly on V and is not realizable over a proper subfield of $\mathbf{F}$. Moreover $\widehat{S}$ fixes a non-degenerate quadratic form on V so that the Frobenius-Schur indicator $ind(V)$ is $+$. (See Section 2.2 and 2.3).

3.4 $\Omega_{2m+1}(3)$

3.4.1 Parameters for $\Omega_{2m+1}(3)$

We now assume the hypothesis and notations in section 3.3 with $L = \Omega_{2m+1}(3), m \geqslant 2$. There are two types of non-singular points in V, namely $+$ and $-$ points. For $\xi \in \{\pm\}$, denote by $\mathfrak{E}_\xi(V)$ the set of all non-singular points of type ξ. For any $\langle x \rangle \in \mathfrak{E}_\xi(V)$, define

$$\Delta(x) = \mathfrak{E}(V) \cap x^\perp \text{ and } \mathit{\Gamma}(x) = \mathfrak{E}(V) \cap (V - x^\perp - \{\langle x \rangle\}).$$

Then P has exactly three orbits $\{\langle x \rangle\}, \Delta(x)$ and $\mathit{\Gamma}(x)$ on $\mathfrak{E}_\xi(V)$. For $\xi = \pm$, we write ξ instead of $\xi 1$ to denote ± 1. Recall the Higman rank 3 parameters defined in section 3.1.

Lemma 3.8 *Let $\xi \in \{\pm\}$ and $\langle x \rangle \in \mathfrak{E}_\xi(V)$. Then*

(i) $|\mathfrak{E}_\xi(V)| = \frac{1}{2}3^m(3^m + \xi)$;

(ii) $k = \frac{1}{2}3^{m-1}(3^m - \xi)$;

(iii) $l = (3^m - \xi)(3^{m-1} + \xi)$;

(iv) $\mu = \lambda = \frac{1}{2}3^{m-1}(3^{m-1} - \xi)$;

(v) $\sqrt{D} = 2.3^{m-1}$;

(vi) $s = 3^{m-1}$;

(vii) $t = -3^{m-1}$.

Proof. Let $\gamma = Q(x)$. We have $\rho_V(x) = sgn(x_V^\perp) = \xi$. Since $q = 3$, by Lemma 2.10, $\mathfrak{E}_\xi(V) = \{\langle v \rangle \mid v \in V_\gamma\}$. Observe that each point in V has $q-1$ representatives. Therefore by Lemma 2.11, $|\mathfrak{E}_\xi(V)| = \frac{1}{2}S(2m+1, x) = \frac{1}{2}(3^{2m} + \xi 3^m) = \frac{1}{2}3^m(3^m + \xi)$, which gives (i). From definition $k = |\Delta(x)| = |\mathfrak{E}_\xi(V) \cap x^\perp| = |\{\langle v \rangle \subseteq x^\perp \mid Q(v) = Q(x)\}|$. By Lemma 2.11 again, $k = \frac{1}{2}S^\xi(2m, \gamma) = \frac{1}{2}S^\xi(2m, \gamma) = \frac{1}{2}(q^{2m-1} - \xi q^{m-1})$. The parameter l can be computed easily by the relation $1 + k + l = |\mathfrak{E}_\xi(V)|$. This proves (ii) and (iii). To compute λ, take $\langle y \rangle \in \Delta(x)$. Then $\rho(y) = \rho(x) = \xi, (x, y) = 0$ and $Q(y) = Q(x)$.

We have $\lambda = |\Delta(x) \cap \Delta(y)| = \frac{1}{2}|\{v \in x^{\perp} \cap y^{\perp} \mid Q(v) = Q(x)\}| = \frac{1}{2}S(2m-1, z)$, where $z \in W := x^{\perp} \cap y^{\perp} = \langle x, y\rangle^{\perp}$ with $Q(z) = Q(x)$. We need to determine $\rho_W(z) = \mathrm{sgn} z_W^{\perp}$. Since $x^{\perp} = \langle y\rangle \perp (y^{\perp} \cap x^{\perp}) = \langle y\rangle \perp W$ and $W = z_W^{\perp} \perp \langle z\rangle$, we deduce that $x^{\perp} = \langle y, z\rangle \perp z_W^{\perp}$, where $\dim z_W^{\perp} = 2m-2, \dim W = 2m-1$ and $\dim\langle y, z\rangle = 2$. Now, as $D\langle y, z\rangle = det(\mathrm{diag}(-\gamma, -\gamma)) = \square$, by Proposition 2.6, $\mathrm{sgn}\langle y, z\rangle = (-)^1 = -$. It follows from Proposition 2.7 that $\mathrm{sgn} z_W^{\perp} = -\mathrm{sgn} x^{\perp} = -\xi$. Thus by Lemma 2.11, $\lambda = \frac{1}{2}(3^{2m-2} - \xi 3^{m-1})$, which is (iv). The remaining parameters follow from Lemma 3.1 and Lemma 3.2. ■

Corollary 3.9 *Let M be a subgroup of G and $\xi \in \{\pm\}$. Suppose that equation (3.1) holds for some $r \in \{s, t\}$, and for some M-orbit $\langle x\rangle M$ with $x \in \mathfrak{E}_{\xi}(V)$. Then*

(i) If $(\xi, r) = (+, s)$ or $(-, t)$ then equation (3.1) has the form

$$c - 2d = \xi 3^m - 1. \tag{3.2}$$

(ii) If $(\xi, r) = (+, t)$ or $(-, s)$ then equation (3.1) has the form

$$(\xi 3^{m-1} + 1)(\xi 3^m - 1 + c - 2d) = 2c \tag{3.3}$$

(iii) If $m \geqslant 2$ then

$$1 + c + d \geqslant \frac{3^m + 1}{2} \geqslant 3^{m-1}. \tag{3.4}$$

Proof. From definitions $c \geqslant 0$ and $d \geqslant 0$. Let $A = 1 + c + d$. Multiply both sides of equation (3.1) by l, we have $(r+1)c = l + \dfrac{drl}{k}$. Subtract $\dfrac{drl}{k}$ from both sides,

$$(r+1)c - \frac{drl}{k} = l. \tag{3.5}$$

(1) If $\xi = +$ and $r = s$, then by Lemma 3.8, we have $r = 3^{m-1}$, $l = (3^m - 1)(3^{m-1} + 1)$, and $\dfrac{l}{k} = 2(3^{m-1} + 1)/3^{m-1}$. From (3.5), $(3^{m-1} + 1)c - 2d(3^{m-1} + 1) = (3^m - 1)(3^{m-1} + 1)$.

Divide both sides by $3^{m-1}+1$, we get $c-2d = 3^m - 1 = \xi 3^m - 1$. Hence $c = 2d+3^m-1$. Thus $A = 3^m + 3d \geqslant 3^m \geqslant \frac{1}{2}(3^m+1)$.

(2) If $\xi = -$ and $r = t$, then by Lemma 3.8, we have $r = -3^{m-1}$, $l = (3^m+1)(3^{m-1}-1)$, and $\dfrac{l}{k} = 2(3^{m-1}-1)/3^{m-1}$. From (3.5) $(-3^{m-1}+1)c+2d(3^{m-1}-1) = (3^m+1)(3^{m-1}-1)$. Divide both sides by $3^{m-1}-1$, we get $c-2d = -3^m-1 = \xi 3^m - 1$. In this case, $2d = 3^m+1+c$. Since $c \geqslant 0$, $2A = 2+2c+3^m+1+c \geqslant 3^m+3 > 3^m+1$.

(3) If $\xi = +$ and $r = t$, then by Lemma 3.8, we have $r = -3^{m-1}$, $l = (3^m-1)(3^{m-1}+1)$, and $\dfrac{l}{k} = 2(3^{m-1}+1)/3^{m-1}$. From (3.5) $(-3^{m-1}+1)c+2d(3^{m-1}+1) = (3^m-1)(3^{m-1}+1)$ or $2c = (3^{m-1}+1)(3^m-1+c-2d) = (\xi 3^{m-1}+1)(\xi 3^m - 1 + c - 2d)$, and $2d(3^{m-1}+1) = (3^{m-1}+1)(3^m-1) + (3^{m-1}-1)c$. As $c \geqslant 0$, $2d(3^{m-1}+1) \geqslant (3^{m-1}+1)(3^m-1)$, or $2d \geqslant 3^m-1$. Now $2A = 2+2c+2d \geqslant 2+2c+3^m-1 \geqslant 3^m+1$.

(4) If $\xi = -$ and $r = s$, then by Lemma 3.8, we have $r = 3^{m-1}$, $l = (3^m+1)(3^{m-1}-1)$, and $\dfrac{l}{k} = 2(3^{m-1}-1)/3^{m-1}$. From (3.5) $(3^{m-1}+1)c-2d(3^{m-1}-1) = (3^m+1)(3^{m-1}-1)$, or equivalently $2c = (3^{m-1}-1)(3^m+1-c+2d) = (\xi 3^{m-1}+1)(\xi 3^m - 1 + c - 2d)$. Therefore $(3^{m-1}+1)c = (3^{m-1}-1)(3^m+1)+2d(3^{m-1}-1)$. As $d \geqslant 0$, $c \geqslant (3^{m-1}-1)(3^m+1)/(3^{m-1}+1)$. Now $A \geqslant 1+(3^{m-1}-1)(3^m+1)/(3^{m-1}+1) = (3^{2m-1}-3^{m-1})/(3^{m-1}+1)$. However $(3^{2m-1}-3^{m-1})/(3^{m-1}+1) - \frac{1}{2}(3^m+1) = (3^m-1+3^{m+1}(3^{m-2}-1))/(2(3^{m-1}+1)) \geqslant 0$. Therefore $A \geqslant \frac{1}{2}(3^m+1)$. Finally $\frac{1}{2}(3^m+1) > \frac{1}{2}3^m > 3^{m-1}$. The proof is completed. ■

3.4.2 Permutation characters of maximal subgroups in $\mathcal{C}$

By Proposition 2.6.2 in [29], $A = I = S \times \langle -1 \rangle$ and $S = \Omega\langle r_\square r_\boxtimes \rangle$. Let $M \in \mathcal{C}(G)$ be a maximal subgroup of G. Let $M_\Xi \in \mathcal{C}(\Xi)$ be such that $M = G \cap M_\Xi$. Then $M_I = M_\Delta = M_\Xi$, and $M_\Omega \leqslant M \leqslant M_I$.

The reducible subgroups $\mathcal{C}_1$

The reducible subgroups $\mathcal{C}_1$ are all the groups M_Ξ of the forms $N_\Xi(W)$, where $dimW = \alpha$, $1 \leqslant \alpha \leqslant \dfrac{n}{2}$ and W is either non-degenerate or totally singular. The corresponding subgroups are of type $O_{\alpha}^{\varepsilon_1}(q) \perp O_{n-\alpha}^{\varepsilon_2}(q)$ or P_α, where $n = dimV, \varepsilon_1 = sgn(W), \varepsilon_2 = sgn(W^{\perp})$, and $(\alpha, \varepsilon_1) \neq (n-\alpha, \varepsilon_2)$. The subgroup M_Ξ is maximal in Ξ except when M_Ξ is of type $O_2^+(3) \perp O_{n-2}(3)$, as it is contained in subgroups of type $O_1(3) \perp O_{n-1}^{\pm}(3)$.

Proposition 3.10 *Assume M is of type $O_{\alpha}^{\varepsilon_1}(3) \perp O_{2m+1-\alpha}^{\varepsilon_2}(3)$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless M is of type $O_1(3) \perp O_{2m}^{\varepsilon_2}(3)$. In this case M is in Table 1.1.*

Proof. As M is of type $O_{\alpha}^{\varepsilon_1}(3) \perp O_{2m+1-\alpha}^{\varepsilon_2}(3)$, there exists a non-degenerate subspace $W \leqslant V$ of dimension α such that $M = N_G(W)$. Put $W_1 = W, W_2 = W^{\perp}, \varepsilon_i = sgnW_i$. Write $X_i = X(W_i)$, where X ranges over the symbols Ω, S and I. By Lemma 4.1.1 in [29], we have $M_I = I_1 \times I_2$ and $\Omega_1 \times \Omega_2 \leqslant M_\Omega$. Without loss of generality, we can suppose that $\dim W_1 = 2b+1$, where $0 \leqslant b < m$. If $b = 0$, then the proposition holds. Assume that $b \geqslant 1$. Then W_1 contains non-singular points of both types. Let $\langle x_i \rangle, i = 1, 2$ be non-singular points in W_1 of different types. By Lemma 2.3, $\langle x_i \rangle \Omega_1 = \langle x_i \rangle I_1, i = 1, 2$. Moreover, as I_2 centralizes $x_i, i = 1, 2$, we have $\langle x_i \rangle M_\Omega = \langle x_i \rangle M_I$, so that $\langle x_i \rangle M_\Omega = \langle x_i \rangle M$. Thus it is sufficient to compute parameters c, d for subgroup M_Ω in L. Let $x \in \{x_1, x_2\}$, $\eta = \rho_{W_1}(x)$ and $\xi = \rho_V(x)$. Since $\langle x \rangle M_\Omega = \mathfrak{E}_\eta(W_1)$, it follows that $c = l(W_1)$ and $d = k(W_1)$, the parameters for $\mathfrak{E}_\eta(W_1)$. By Lemma 3.8, $d = \frac{1}{2}3^{b-1}(3^b - \eta),\ c = (3^b - \eta)(3^{b-1} + \eta)$, and so $c - 2d = \eta 3^b - 1$. Suppose that equation (3.1) holds for some $r = s, t$ and any M-orbits on $\mathfrak{E}_\xi(V)$. If (3.2) holds then $\eta 3^b - 1 = \xi 3^m - 1$. This implies that $b = m$, a contradiction. Thus (3.3) holds. Then $(3^{m-1} + \xi)(3^m - 2\xi + \xi\eta 3^b) = 2(3^b - \eta)(3^{b-1} + \eta)$. Observe that $m - 1 \geqslant b$. Assume first that $m - 1 = b$ and $\xi = -\eta$. We have $3^{m-1} + \xi = 3^b - \eta$ and $3^m - 2\xi + \xi\eta 3^b = 3^{b+1} + 2\eta - 3^b = 2(3^b + \eta)$. Clearly $2(3^b + \eta) > 2(3^{b-1} + \eta)$, and hence equation (3.3) does not hold in this case. Next assume that $m - 1 = b$ and $\xi = \eta$. Then $(3^{m-1} + \xi)(3^m - 2\xi + \xi\eta 3^b) = (3^b + \xi)(3^{b+1} - 2\xi + 3^b) = 2(3^b + \eta)(2.3^b - \eta) > 2(3^{b-1} + \eta)(3^b - \eta)$, a contradiction. Finally assume that $m - 1 > b$ so that $m - 1 \geqslant b + 1$. Then $3^{m-1} + \xi \geqslant 3^{b+1} + \xi > 3^b - \eta$ and $3^m - 2\xi + \xi\eta 3^b \geqslant 9.3^b - 2\xi - 3^b = 8.3^b - 2\xi > 2(3^{b-1} + \eta)$. Multiplying these two inequalities side by side will lead to a contradiction. Thus equation (3.3) does not hold. Therefore $b = 0$ and so W is a point or a hyperplane. ■

Let W be a totally singular subspace of V of dimension α. By Witt's Lemma, we can assume that W has a basis $\{e_1, \ldots, e_\alpha\}$, where the vectors e_i are taken from a standard basis of V. Let

$$Y = \langle f_1, \ldots, f_\alpha \rangle, \text{ and } X = \langle e_{\alpha+1}, \ldots, e_m, f_{\alpha+1}, \ldots, f_m, a \rangle.$$

Then $V = (W \bigoplus Y) \perp X$, $W^\perp = W \perp X$. Finally let

$$U = C_{I(V)}(W, W^\perp/W, V/W^\perp), \text{ and } N = N_{I(V)}(W, Y, X).$$

The relation among these groups is given in the following lemmas.

Lemma 3.11 ([29], Lemma 4.1.9). *Assume that $n = 2m$ is even and that case $\mathbf{U}, \mathbf{S}$ or $\mathbf{O}^+$ holds. Let $\beta = \{e_1, \cdots, e_m, f_1, \cdots, f_m\}$ be the corresponding standard basis. Let $W_1 = \langle e_1, \cdots, e_m \rangle, W_2 = \langle f_1, \cdots, f_m \rangle$, and $T_0 = N_{I(V)}(W_1, W_2)$, where $V = \langle \beta \rangle$. Then*
(i) $T_0 \cong GL_m(q^u)$ and T_0 acts naturally on W_1.
(ii) As T_0-modules we have $W_2 \cong W_1^{\alpha *}$, where α is an element of order u in $Aut(\mathbf{F})$.
(iii) $T_0 \cap \Omega(V) = \{x \in T_0 \mid det_{W_1}(x) \in (\mathbf{F}^*)^z\}$, where $z = q+1, 1, 2$ in cases $\mathbf{U}, \mathbf{S}$ and $\mathbf{O}^+$, respectively.

Lemma 3.12 ([29], Lemma 4.1.12). *Let W be a totally singular subspace of V. Keeping the notations above, we have:*
(i) $M_I = U : N$;
(ii) $N = T_0 \times I(X)$, where $GL_\alpha(q^u) \simeq T_0 \leqslant I(W \oplus Y)$, and T_0 acts naturally on W; and as T_0-modules we have $Y \cong W$;
(iii) U is a p-group and $U \leqslant \Omega(V)$.

It follows from (i) and (iii) of Lemma 3.12 that $M_\Omega = U(N \cap \Omega(V))$.

Proposition 3.13 *Assume M is of type P_α. Then M has at most two orbits in $\mathfrak{E}_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7 and so M is in Table 1.1.*

Proof. We will show that M_Ω has at most two orbits on $\mathfrak{E}(V)$, and hence we deduce that M also has at most two orbits on $\mathfrak{E}(V)$. Assume the notation above and denote by f the associated bilinear form of Q. From definition M stabilizes a totally singular subspace W of dimension α. Let Y and X be defined as above. By Proposition 2.8, X has a basis $\beta_X = \{x_1, \ldots, x_s\}$, with $s = 2m+1-2\alpha = 2(m-\alpha)+1$, such that $[f_{\downarrow X}]_{\beta_X} = \lambda \mathbf{I}_s$, where $D(X) \equiv \lambda (\text{mod } (\mathbf{F}^*)^{\mathbf{2}})$. Let $\beta = \{e_1, \ldots, e_\alpha, x_1, \ldots, x_s, f_1, \ldots, f_\alpha\}$. Let $x \in X$ be a non-singular point with $\xi = \rho_X(x)$. As $sgn(W \oplus Y) = +$, $\xi = \rho_V(x)$. If $\alpha < m$, then $s = dim(X) = 2(m-\alpha)+1 \geqslant 3$, so that X contain both plus and minus points. Otherwise, X has no minus points. As $\Omega(X) \leqslant M_\Omega$, and $U \leqslant M_\Omega$, we have $x(\Omega(X)U) \subseteq xM_\Omega$. For any $v \in x\Omega(X)$, $w \in W$, we will show that there exists $u \in U$ such that $vu = v + w$, which implies that $|\langle x\rangle\Omega(X)U| = |\mathfrak{E}_\xi(X)|.|W|$. Thus

$$|\langle x\rangle M_\Omega| \geqslant \frac{1}{2}|x\Omega(X)U| = |W|.|\mathfrak{E}_\xi(X)| = 3^\alpha.\frac{1}{2}3^{m-\alpha}(3^{m-\alpha}+\xi)) = \frac{1}{2}3^m(3^{m-\alpha}+\xi).$$

Therefore

$$|\langle x\rangle M_\Omega| \geqslant \frac{1}{2}3^m(3^{m-\alpha}+\xi). \tag{3.6}$$

Let $\widehat{B} = [f]_\beta$. Then

$$\widehat{B} = \begin{pmatrix} 0 & 0 & I_\alpha \\ 0 & \lambda I_s & 0 \\ I_\alpha & 0 & 0 \end{pmatrix}.$$

We have $[v]_\beta = (0, a, 0)$ and $[w]_\beta = (b, 0, 0)$, where a, b are row vectors in $\mathbf{F}^{\mathbf{s}}$ and $\mathbf{F}^\alpha$, respectively. Since v is non-singular, v is non-zero. Choose $B \in M_{s\times\alpha}(\mathbf{F})$ such that

$aB = b$. Let

$$C = -\lambda^{-1}B^t, A + A^t = -\lambda^{-1}B^tB = -\lambda CC^t,$$

and

$$u = \begin{pmatrix} I_\alpha & 0 & 0 \\ B & I_s & 0 \\ A & C & I_\alpha \end{pmatrix}.$$

Then $u\widehat{B}u^t = \widehat{B}$, u centralizes $W, W^\perp/W$ and $V/W^\perp$. Thus $u \in U$ and $vu = v + w$.

Let $z = \eta e_1 + f_1 \in V$, where $\eta = Q(x)$. Then $Q(z) = Q(x)$ so that $\langle x \rangle$ and $\langle z \rangle$ belong to the same Ω-orbit of non-singular points in V. Let $T = \frac{1}{2}T_0$. By Lemma 3.11, $SL_\alpha(3) \simeq T \leqslant N \cap \Omega \leqslant M_\Omega$, and so $TU \leqslant M_\Omega$, where T_0, N are as in Lemma 3.12. Thus $z(TU) \subseteq zM_\Omega$. We find the stabilizer $(TU)_z$ in TU of vector z. For any $g \in TU$, there exist $h \in T, u \in U$ such that $g = hu$. We have

$$[h]_\beta = \begin{pmatrix} D & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & D^* \end{pmatrix}, \text{ and } [u]_\beta = \begin{pmatrix} I_\alpha & 0 & 0 \\ B & I_s & 0 \\ A & C & I_\alpha \end{pmatrix},$$

where $D = (d_{ij}) \in SL_\alpha(3)$, D^* its inverse transpose, and

$$C = -\lambda^{-1}B^t, A + A^t = -\lambda^{-1}B^tB = -\lambda CC^t.$$

Thus $[g]_\beta = [hu]_\beta = [h]_\beta[u]_\beta$. Suppose $g \in (TU)_z$. Write

$$E_1 = (\underbrace{\eta, 0, \ldots, 0}_{\alpha}) \text{ and } F_1 = (\underbrace{1, 0, \ldots, 0}_{\alpha}).$$

Then $[z]_\beta = (E_1, 0, F_1)$. As $zg = z$, we have

$$(E_1, 0, F_1)\begin{pmatrix} D & 0 & 0 \\ B & I_s & 0 \\ D^*A & D^*C & D^* \end{pmatrix} = (E_1, 0, F_1),$$

or $(E_1D + F_1D^*A, F_1D^*C, F_1D^*) = (E_1, 0, F_1)$, hence

$$\begin{cases} F_1D^* & = & F_1 \\ F_1D^*C & = & 0 \\ E_1D + F_1D^*A & = & E_1 \end{cases}$$

Since $F_1D^* = F_1$,

$$D^* = \begin{pmatrix} 1 & 0 \\ b & D_1^* \end{pmatrix}, D = \begin{pmatrix} 1 & -b^tD_1 \\ 0 & D_1 \end{pmatrix}, \text{ and } D^{-1} = \begin{pmatrix} 1 & b^t \\ 0 & D_1^{-1} \end{pmatrix},$$

where $D_1 \in SL_{\alpha-1}(3)$ and b is a column vector of size $\alpha-1$. As $F_1D^*C = 0$ and $F_1D^* = F_1$, we have $F_1C = 0$. Hence

$$C = \begin{pmatrix} 0 & 0 \\ c_0 & C_1 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a \\ a_0 & A_1 \end{pmatrix},$$

where $C_1 \in M_{\alpha-1,s-1}(3)$, c_0 is a column vector of size $\alpha - 1$, $A_1 \in M_{\alpha-1}(3), a_{11} \in \mathbf{F}$ and a, a_0 are row, column vectors, respectively, of size $\alpha - 1$. Now as

$$E_1D + F_1D^*A = E_1D + F_1A = E_1,$$

we have

$$(\xi,0)\begin{pmatrix}1 & -b^tD_1\\ 0 & D_1\end{pmatrix}+(1,0)\begin{pmatrix}a_{11} & a\\ a_0 & A_1\end{pmatrix}=(\eta,0).$$

It follows that $(\eta+a_{11}, a-\eta b^tD_1)=(\eta,0)$. Therefore $a_{11}=0$ and $a=\eta b^tD_1$.

Finally as $A+A^t=-\lambda CC^t$, we have

$$\begin{pmatrix}0 & a_0^t+\eta b^tD_1\\ a_0+\eta D_1^tb & A_1+A_1^t\end{pmatrix}=-\lambda^{-1}\begin{pmatrix}0 & 0\\ 0 & c_0c_0^t+C_1C_1^t\end{pmatrix},$$

hence $a_0=-\xi D_1^tb$, $A_1+A_1^t=-\lambda(c_0c_0^t+C_1C_1^t)$ and

$$A=\begin{pmatrix}0 & \eta b^tD_1\\ -\eta D_1^tb & A_1\end{pmatrix}.$$

In summary, for any $g\in(TU)_z$, we have

$$[g]_\beta=\begin{pmatrix}D & 0 & 0\\ B & I_s & 0\\ D^*A & D^*C & D^*\end{pmatrix}=\begin{pmatrix}D & 0 & 0\\ 0 & I_s & 0\\ 0 & 0 & D^*\end{pmatrix}\begin{pmatrix}I_\alpha & 0 & 0\\ B & I_s & 0\\ A & C & I_\alpha\end{pmatrix},$$

where

$$D=\begin{pmatrix}1 & -b^tD_1\\ 0 & D_1\end{pmatrix}, D^*=\begin{pmatrix}1 & 0\\ b & D_1^*\end{pmatrix}, C=\begin{pmatrix}0 & 0\\ c_0 & C_1\end{pmatrix}, A=\begin{pmatrix}0 & \xi b^tD_1\\ -\xi D_1^tb & A_1\end{pmatrix},$$

$$B=-\lambda C^t\in M_{s,\alpha}(3),$$

with $C_1\in M_{\alpha-1,s-1}(3), A_1\in M_{\alpha-1}(3), b, c_0\in M_{\alpha-1,1}(3), A_1+A_1^t=-\lambda^{-1}(c_0c_0^t+C_1C_1^t)$, $A\in M_\alpha(3), D, D^*\in SL_\alpha(3), C\in M_{\alpha,s}(3)$. We see that the subgroup of $SL_\alpha(3)$ generated by all matrices D is isomorphic to $U_0 : SL_{\alpha-1}(3)$, where U_0 is an elementary abelian subgroup of order $3^{\alpha-1}$. Given such D, there are $3^{\alpha-1+(\alpha-1)(s-1)}$ choices for C and $3^{\frac{1}{2}(\alpha-1)(\alpha-2)}$ choices for A. Therefore

$$|(TU)_z| = 3^{\alpha-1}.|SL_{\alpha-1}(3)|.3^{\alpha-1+(\alpha-1)(s-1)+\frac{1}{2}(\alpha-1)(\alpha-2)}.$$

Since $|U| = 3^{\alpha.s+\frac{1}{2}\alpha(\alpha-1)}$, we have $|z(TU)| = |TU : (TU)_z| = 3^{s+\alpha-1}(3^\alpha - 1)$. Thus

$$|\langle z\rangle M_\Omega| \geqslant \frac{1}{2}3^{s+\alpha-1}(3^\alpha - 1). \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$|\mathfrak{E}_\xi(V)| \geqslant |\langle x\rangle M_\Omega| + |\langle z\rangle M_\Omega| \geqslant \frac{1}{2}3^m(3^{m-\alpha}+\xi) + \frac{1}{2}3^{2m-\alpha}(3^\alpha - 1) = \frac{1}{2}3^m(3^m+\xi) = |\mathfrak{E}_\xi(V)|.$$

Therefore $|\langle x\rangle M_\Omega| = |\langle x\rangle\Omega(X)U|$, $|\langle z\rangle M_\Omega| = |\langle z\rangle M_\Omega U|$, so that $\mathfrak{E}_\xi(V) = \langle x\rangle M_\Omega \cup \langle z\rangle M_\Omega$. Hence M_Ω has at most two orbits on $\mathfrak{E}_\xi(V)$. Clearly as $M_\Omega \leqslant M$, $\langle x\rangle M_\Omega \subseteq \langle x\rangle M$ and similarly $\langle z\rangle M_\Omega \subseteq \langle z\rangle M$. Since $\mathfrak{E}_\xi(V) = \langle x\rangle M_\Omega \cup \langle z\rangle M_\Omega$ and $\langle x\rangle M \cup \langle z\rangle M \subseteq \mathfrak{E}_\xi(V)$, it implies that $\mathfrak{E}_\xi(V) = \langle x\rangle M \cup \langle z\rangle M$. Thus M has at most two orbits on $\mathfrak{E}_\xi(V)$. ■

The imprimitive subgroups $\mathcal{C}_2$

A *subspace decomposition* $\mathcal{D} = \{V_1, \ldots, V_b\}$ of V is a set of subspaces $V_1, \ldots, V_b$ of V with $b \geqslant 2$ such that $V = V_1 \oplus V_2 \oplus \cdots V_b$. Let $\mathfrak{G}$ be a subgroup of $GL(V)$. The *stabilizer* in $\mathfrak{G}$ of $\mathcal{D}$ is the group $N_{\mathfrak{G}}\{V_1, \ldots, V_b\}$, which is the subgroup of $\mathfrak{G}$, permuting the spaces V_i amongst themselves and denoted by $\mathfrak{G}_{\mathcal{D}}$. The *centralizer in* $\mathfrak{G}$ of $\mathcal{D}$, is the group $\mathfrak{G}_{(\mathcal{D})} = N_{\mathfrak{G}}(V_1, \ldots, V_b)$, which is a subgroup of $\mathfrak{G}$ fixing each V_i. We also define $\mathfrak{G}^{\mathcal{D}} = \mathfrak{G}_{\mathcal{D}}/\mathfrak{G}_{(\mathcal{D})}$. If the spaces V_i in the subspace decomposition $\mathcal{D}$ all have the same dimension α, then $\mathcal{D}$ is called an *α-decomposition.* If the $V_i's$ are non-degenerate and pairwise orthogonal, then $\mathcal{D}$ is said to be *non-degenerate.* For any vector $v \in V$, v can be written in the form $v = v_1 + v_2 + \cdots + v_b$, where $v_i \in V_i$. We define the $\mathcal{D} - length$ of v to be the number of non-zero vectors v_i appearing in v, and denote by $\mathcal{D}_b^k$, the number of all points of $\mathcal{D}-$length $k, 1 \leqslant k \leqslant b$. The members of $\mathcal{C}_2(\mathit{\Gamma O}_n(q))$ are the stabilizers in $\mathit{\Gamma O}_n(q)$ of α-decomposition $\mathcal{D}$ of V such that $\mathcal{D}$ is non-degenerate and if $\alpha = 1$ then $q = p$, a prime.

Lemma 3.14 ([29], Proposition $4.2.11, 4.2.15, 4.2.15$). *Let M_Ω be the stabilizer in $\Omega(V)$ of a non-degenerate α-decomposition $\mathcal{D}$ and $n = b\alpha$. Then*
(i) If $\alpha > 1$, then $M_\Omega \cong \Omega(V)_{(\mathcal{D})}.S_b$.
(ii) If $\alpha = 1$, and $q \equiv \pm 3 \pmod 8$, then $M_\Omega \cong \Omega(V)_{(\mathcal{D})}.A_n$.

Lemma 3.15 *Assume that $\mathcal{D}$ is a 1-decomposition of V with q odd. For $1 \leqslant k \leqslant n$, $|\mathcal{D}_n^k| = (q-1)^{k-1}\binom{n}{k}$.*

Proof. Without loss of generality, we can assume that V has an orthonormal basis $\beta = \{v_1, \cdots, v_n\}$. If $v \in V$ has $\mathcal{D}$-length k then v is a linear combination of a set of k basic vectors taken from the basis β, with coefficients in $\mathbf{F}_q^*$. Clearly, there are $\binom{n}{k}$ choices for k-sets, and for each k-set, there are $(q-1)^{k-1}$ points of length k. The result follows. ■

Lemma 3.16 *Assume G is nearly simple primitive rank 3 of type $\Omega_n^\varepsilon(3)$ with $\varepsilon \in \{\circ, \pm\}$ and M is of type $O_1(3) \wr S_n$. Let $\beta = \{x_1, \ldots, x_n\}$ be an orthogonal basis for V. Then*
(1) If $z = x_1$ then $d_1 = d_z = n - 1$ and $c_1 = c_z = 0$;
(2) If $z = x_1 + x_2$ then $d_2 = d_z = n^2 - 5n + 7$ and $c_2 = c_z = 4n - 8$.

Proof. By multiplying a suitable non-zero constant to the quadratic form Q, we can assume that V has an orthonormal basis $\beta = \{x_1, \ldots, x_n\}$. Setting $r_i = r_{x_i}$, the reflection along vector x_i. Then we have

$$I(V)_{(\mathcal{D})} = \langle r_i | 1 \leqslant i \leqslant n \rangle \cong 2^n, \text{ and } \Omega(V)_{(\mathcal{D})} = \langle r_i r_j | 1 \leqslant i, j \leqslant n \rangle \cong 2^{n-1}.$$

For $1 \leqslant i \neq j \leqslant n$, we see that $r_{x_i-x_j}$ permutes $\{x_i, x_j\}$, and fixes x_t for any $t \notin \{i,j\}$ and so $r_{x_i-x_j}$ acts as a transposition (i,j). Thus if we denote by J, the group generated by all reflections $r_{x_i-x_j}$, where $1 \leqslant i \neq j \leqslant n$, then $J \cong S_n$ and hence J_1, the subgroup of J generated by $r_{x_i-x_j}r_{x_r-x_s}$, with $i \neq j, r \neq s$ is isomorphic to A_n. By Lemma 3.14(ii), $M_\Omega = \Omega(V)_{(\mathcal{D})}J_1$. For any $1 \leqslant i \neq j \leqslant n$, as $(x_i, x_j) = 0$, r_j fixes x_i. Hence $\Omega(V)_{(\mathcal{D})}$ leaves invariant point $\langle x_1 \rangle$, and $\langle x_1 + x_2 \rangle \Omega(V)_{(\mathcal{D})} = \{\langle x_1 + x_2 \rangle, \langle x_1 - x_2 \rangle\}$, as $(x_1 + x_2)r_2r_3 = x_1 - x_2$. By (4.2.17) in [29], we have $M_I = M_\Omega \langle r_3, r_{x_3-x_4} \rangle$. Thus $\langle x \rangle M_\Omega = \langle x \rangle M_I$ for any $x \in \{x_1, x_1 + x_2\}$, so that it suffices to compute the parameters for M_Ω in L. Now, since $n \geqslant 5$, A_n acts transitively on the set $\{1, 2, \cdots, n\}$. Thus $\langle x_1 \rangle M_\Omega = \{\langle x_1 \rangle, \cdots, \langle x_n \rangle\}$. Hence $1+c_1+d_1 = n$. Moreover, as $(x_i, x_1) = 0$ for any $i > 1$, we have $d_1 = |x_1^\perp \cap \langle x_1 \rangle M_\Omega| = |\{\langle x_2 \rangle, \cdots, \langle x_n \rangle\}| = n - 1$, and so $c_1 = 0$. Similarly, as A_n acts doubly transitively on $\{1, \cdots, n\}$, we have $\langle x_1 + x_2 \rangle M_\Omega = \langle x_1 + x_2 \rangle \Omega(V)_{(\mathcal{D})}J_1 = \{\langle x_1 + x_2 \rangle, \langle x_1 - x_2 \rangle\}J_1$. Thus $1 + c_2 + d_2 = |\langle x_1 + x_2 \rangle M_\Omega| = \mathcal{D}_n^2 = n(n-1)$, by Lemma 3.15. For any $\langle v \rangle \in \langle x_1 + x_2 \rangle^\perp \cap \langle x_1 + x_2 \rangle M_\Omega$, $\langle v \rangle = \langle x_i \pm x_j \rangle$ for some $i \neq j \in \{1, \cdots, n\}$ and $(v, x_1 + x_2) = 0$. Clearly, $\langle v \rangle$ is generated by $x_1 - x_2$ or $x_i \pm x_j$ for some $i < j \in \{3, \cdots, n\}$. By Lemma 3.15 again, $d_2 = |\langle x_1 + x_2 \rangle^\perp \cap \langle x_1 + x_2 \rangle M_\Omega| = \mathcal{D}_{n-2}^2 + 1 = n^2 - 5n + 7$, and so $c_2 = 4n - 8$. ■

Proposition 3.17 *Assume M is of type $O_1(3) \wr S_n$, with $n = 2m + 1$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $(n, \xi, r) = (5, +, t), (7, +, t)$ or $(5, -, s)$, in which cases M has 2 orbits on $\mathfrak{E}_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7, and hence M is in Table 1.1.*

Proof. Retain the notations in previous lemma. By Proposition 2.6, $sgn(x_1^\perp) = (-1)^m$ and $sgn(x_1 + x_2)^\perp = (-1)^{m+1}$, as the discriminant of the corresponding subspaces is square or non-square respectively. When m is even x_1 is a plus vector and $x_1 + x_2$ is a minus vector and vice versa when m is odd. Let $x \in \{x_1, x_1 + x_2\}$. We consider the following cases:

(i) $\langle x\rangle$ is a plus point. If m is even then we choose $x = x_1$. By Lemma 3.16 $d = d_1 = n-1, c = c_1 = 0$. Then $k = -rd$, so r must be t and so $3^m - 1 = 4m$. This equation holds only when $m = 2$ and hence $n = 5$. If m is odd then choose $x = x_1 + x_2$, and hence $d = d_2 = (n-2)(n-3)+1, c = c_2 = 4n-8$ by Lemma 3.16 again. Then $c - 2d = 14n - 2n^2 - 22$. As $n \geqslant 5$, $2n^2 + 22 > 14n$ so that $c - 2d < 0$. Therefore, equation (3.2) cannot hold. If equation (3.3) holds then $(3^{m-1}+1)(3^m - 1 + c - 2d) = 2c$. It follows that $(3^{m-1}+1)(3^m-1+14n-2n^2-22) = 8(2m-1)$. If $m \geqslant 5$, then $3^{m-1}+1 > 8(2m-1)$, hence this equation cannot hold. For $2 \leqslant m \leqslant 4$, the equation occurs only when $m = 3$. Thus equation (3.1) holds only when $r = t$ and $m = 3$ or $n = 7$.

(ii) $\langle x\rangle$ is a minus point. If m is odd then $x = x_1$ and $d = n-1, c = 0$. Then $k = -rd$. It follows that $r = t$, and hence $3^m + 1 = 4m$. Since $m \geqslant 2, 3^m + 1 > 4m$, so that this equation cannot hold. If m is even, $x = x_1 + x_2$ and $d = (n-2)(n-3)+1, c = 4n-8$. We have $2d - c = 2n^2 + 22 - 14n$. If equation (3.3) holds then $(3^{m-1}-1)(3^m+1+2d-c) = 2c$, hence $(3^{m-1}-1)(3^m+1+2n^2+22-14n) = 8(2m-1)$. Since $2n^2+22-14n > 0$, $(3^{m-1}-1)(3^m+1+2n^2+22-14n) > (3^{m-1}-1)(3^m+1) > 8(2m-1)$ for any $m \geqslant 3$, and when $m = 2$, $(3^{m-1}-1)(3^m+1+2n^2+22-14n) = 24 = 8(2m-1)$. If equation (3.2) holds then $2d - c = 2n^2+22-14n = 3^m+1$. We can check that $3^m + 1 > 2n^2 + 22 - 14n$ for any $m \geqslant 2$. Thus equation (3.1) holds only when $m = 2$ or $n = 5$ and $r = s$.

To finish the proof, we need to verify that when these cases happen then equation (3.1) also holds for all $\langle x\rangle \in \mathfrak{E}_\xi(V)$. In view of Corollary 3.7, we will show that there are only two orbits of non-singular points of specified types. Firstly, suppose that $n = 5$. Then $m = 2$, and $|\mathfrak{E}_\xi| = \frac{1}{2}3^m(3^m+\xi)$. In this case, $\langle x_1\rangle M_\Omega$ and $\langle x_1+x_2+x_3+x_4\rangle M_\Omega$ are two orbits of plus points with orbit sizes 5 and $2^3\binom{5}{4} = 40$, respectively. However, $\mathfrak{E}_+(V) = \frac{1}{2}3^2(3^2+1) = 45 = 5 + 40$. Hence there are only two orbits of plus points. Similarly, $\langle x_1+x_2\rangle M_\Omega$ and $\langle x_1+x_2+x_3+x_4+x_5\rangle M_\Omega$ are two orbits of minus points with orbit sizes $2\binom{5}{2} = 20$ and $2^4\binom{5}{5} = 16$, respectively. Since $|\mathfrak{E}_-(V)| = \frac{1}{2}3^2(3^2-1) =$

$36 = 20 + 16$, there are exactly two orbits of minus points. Finally, suppose that $n = 7$, and $\xi = +$. Then $m = 3$ and $\mathfrak{E}_+(V) = \frac{1}{2}3^3(3^3 + 1) = 378$. In this case, $\langle x_1 + x_2 \rangle M_\Omega$ and $\langle x_1 + x_2 + x_3 + x_4 + x_5 \rangle M_\Omega$ are two orbits of plus points with orbit sizes $2\binom{7}{2} = 42$ and $2^4\binom{7}{5} = 336$. Clearly, as $336 + 42 = 378$, these are only two orbits of plus points. This completes the proof. ■

We next consider the case when $\alpha > 1$. Since $\dim V$ is odd, it follows that α and b are both odd. Write $\alpha = 2a + 1, b = 2b_1 + 1$.

Proposition 3.18 *Assume M is of type $O_\alpha(3) \wr S_b$, with $\alpha > 1$ odd. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. We can assume that V has an orthonormal basis which is the union of orthonormal bases of all V_i. Let $N = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_b \leqslant M_\Omega$. By Lemma 4.2.8 in [29], we have $M_I = I_1 \wr S_b$. Thus $\Omega_1 \wr S_b \leqslant M_\Omega \leqslant M_I$. Since $\alpha > 1$ is odd, $\alpha \geqslant 3$ and so V_1 contains both plus and minus points. Let $x_\xi \in V_1$ be a non-singular vector of type $\xi \in \{\pm\}$. Clearly $\langle x_\xi \rangle \Omega_1 \wr S_b = \langle x_\xi \rangle I_1 \wr S_b$, we conclude that $\langle x_\xi \rangle M_\Omega = \langle x_\xi \rangle M_I = \langle x_\xi \rangle N.S_b$. Thus we only need to compute the parameter for M_Ω in L. Since $S_b \leqslant M_\Omega$ permutes the $V_i's$, and Ω_i centralizes V_1, for all $i > 1$,

$$\langle x_\xi \rangle M_\Omega = \langle x_\xi \rangle N.S_b = \langle x_\xi \rangle \Omega_1.S_b = \mathfrak{E}_\xi(V_1)S_b = \cup_{i=1}^{b} \mathfrak{E}_\xi(V_i).$$

Thus $A = |\langle x_\xi \rangle M_\Omega| = b.\frac{1}{2}3^a(3^a + \xi)$ by Lemma 3.8(i). Hence $A \leqslant \frac{1}{2}.b.3^a(3^a + 1)$. In view of inequality (3.4), it suffices to show that $\frac{1}{2}3^m \geqslant \frac{1}{2}b.3^a(3^a + 1)$. Since $m = ba + b_1$, this inequality is equivalent to $3^{ba+b_1} \geqslant b.3^a.(3^a + 1)$. As $b \geqslant 3$ and $a \geqslant 1$, $3^{ba} \geqslant 3^{3a} = 3^a.3^{2a} \geqslant 3.3^{2a} > 3^{2a} + 3^a$. Now, if we can prove that $3^{b_1} = 3^{\frac{b-1}{2}} \geqslant b$, then clearly $3^{ba}3^{b_1} \geqslant b.3^a(3^a + 1)$, and we are done. To show that $3^{b_1} \geqslant b$, we will argue by induction on $b \geqslant 3$. When $b = 3$ then $3^{\frac{b-1}{2}} = 3 \geqslant 3 = b$. Suppose that $3^{\frac{b-1}{2}} \geqslant b$. Then $3^{\frac{(b+1)-1}{2}} = 3^{\frac{b-1}{2}}\sqrt{3} \geqslant b.\sqrt{3}$, by induction assumption. We have $3b^2 = b^2 + 2b^2 \geqslant b^2 + 6b > b^2 + 2b + 1 = (b+1)^2$, as $b \geqslant 3$. Thus $b.\sqrt{3} \geqslant b + 1$. Hence $3^{\frac{b+1-1}{2}} \geqslant b + 1$. The result follows. ■

The field extension subgroups $\mathcal{C}_3$

Let $\mathbf{F}_\sharp$ be a field extension of $\mathbf{F} = \mathbf{F}_3$ of degree α, where α is a prime divisor of $n = dimV$. Then V acquires the structure of an $\mathbf{F}_\sharp$-vector space in a natural way. Write $V_\sharp$ for V regarded as a vector space over $\mathbf{F}_\sharp$. Denote by T the trace map from $\mathbf{F}_\sharp$ to $\mathbf{F}$. If $Q_\sharp$ is a quadratic form on $(V_\sharp, \mathbf{F}_\sharp)$ then $Q = TQ_\sharp$ is a quadratic form on $(V, \mathbf{F})$. Write $f_\sharp$ for the associated bilinear form of $Q_\sharp$. Denote by $N = N_{\mathbf{F}_\sharp/\mathbf{F}}$ the norm map of $\mathbf{F}_\sharp$ over $\mathbf{F}$. Let μ, ν be the generators for $\mathbf{F}_\sharp^*$ and $Gal(\mathbf{F}_\sharp/\mathbf{F})$, respectively. Also the trace map from $\mathbf{F}_\sharp$ to $\mathbf{F}$ defines a non-degenerate bilinear form on $\mathbf{F}_\sharp$. Let $Q_T : \mathbf{F}_\sharp \to \mathbf{F}$ be a map defined by $Q_T(x) = -T(x^2)$ for $x \in \mathbf{F}_\sharp$. Then Q_T is a quadratic form on $\mathbf{F}_\sharp$ and $f_T(x, y) = T(xy)$ is the bilinear form associated to Q_T. Then $(\mathbf{F}_\sharp, Q_T, \mathbf{F})$ is an orthogonal geometry.

Lemma 3.19 *Let $\beta_T = \{\zeta_1, \zeta_2, \cdots, \zeta_\alpha\}$ be an $\mathbf{F}$-basis of $\mathbf{F}_\sharp = \mathbf{F}_{p^\alpha}$, where $p = 3$, and $v_\sharp \in V_\sharp$ be such that $f_\sharp(v_\sharp, v_\sharp) = \lambda \in \mathbf{F}_\sharp^*$. Then*
(i) $D(\mathbf{F}_\sharp) \equiv det(f_{\beta_T}) \equiv (-1)^{\alpha-1} (mod\ (\mathbf{F}^*)^2)$;
(ii) $span_{\mathbf{F}_\sharp}(v_\sharp)$ is a non-degenerate α-subspace in V with discriminant $D(\mathbf{F}_\sharp)N(\lambda)$.

Proof. From definition, we have

$$f_{\beta_T} = \begin{pmatrix} T(\zeta_1^2) & T(\zeta_1\zeta_2) & \cdots & T(\zeta_1\zeta_\alpha) \\ T(\zeta_2\zeta_1) & T(\zeta_2^2) & \cdots & T(\zeta_2\zeta_\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ T(\zeta_\alpha\zeta_1) & T(\zeta_\alpha\zeta_2) & \cdots & T(\zeta_\alpha^2) \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_\alpha \\ \zeta_1^p & \zeta_2^p & \cdots & \zeta_\alpha^p \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{p^{\alpha-1}} & \zeta_2^{p^{\alpha-1}} & \cdots & \zeta_\alpha^{p^{\alpha-1}} \end{pmatrix}$$

and $E = diag(\lambda, \lambda^p, \cdots, \lambda^{p^{\alpha-1}})$. As $T(a) = \sum_{i=0}^{\alpha-1} a^{p^i}$ for any $a \in \mathbf{F}_\sharp$, $X^t X = f_{\beta_T}$, hence $det(f_{\beta_T}) = det(X^t X) = det(X)^2$. Since $det(f_{\beta_T}) \in \mathbf{F}^*$ and $det X \in \mathbf{F}_\sharp$, if α is odd then clearly $det X \in \mathbf{F}^*$, as $\mathbf{F}_\sharp$ does not have any subfield of degree 2 over $\mathbf{F}$. Thus $(det X)^2 \in (\mathbf{F}^*)^2 = \{1\}$, so $det(f_{\beta_T}) = (det X)^2 = 1$. Now suppose that $\alpha = 2$. Let ζ be a root of $x^2 - x - 1$ in $\overline{\mathbf{F}}$, and let $\eta = \zeta + 1$. Then $\eta^2 = -1, \mathbf{F}_\sharp = \mathbf{F}(\eta)$ and $T(\eta) = 0$. Choose $\beta_T = \{1, \eta\}$. Then

$$f_{\beta_T} = \begin{pmatrix} T(1) & T(\eta) \\ T(\eta) & T(\eta^2) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Hence $det(f_{\beta_T}) = -1$. This proves (i). Let $W = \langle v_\sharp \rangle_{\mathbf{F}_\sharp}$ and $\beta = \{\zeta_1 v_\sharp, \zeta_2 v_\sharp, \cdots, \zeta_\alpha v_\sharp\}$. As

$$(\zeta_i v_\sharp, \zeta_j v_\sharp) = T(f_\sharp(\zeta_i v_\sharp, \zeta_j v_\sharp)) = T(\zeta_i \zeta_j f_\sharp(v_\sharp, v_\sharp)) = T(\lambda \zeta_i \zeta_j),$$

we have

$$f_\beta = \begin{pmatrix} T(\lambda\zeta_1^2) & T(\lambda\zeta_1\zeta_2) & \cdots & T(\lambda\zeta_1\zeta_\alpha) \\ T(\lambda\zeta_2\zeta_1) & T(\lambda\zeta_2^2) & \cdots & T(\lambda\zeta_2\zeta_\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ T(\lambda\zeta_\alpha\zeta_1) & T(\lambda\zeta_\alpha\zeta_2) & \cdots & T(\lambda\zeta_\alpha^2) \end{pmatrix}.$$

Obviously $X^t E X = f_\beta$. Therefore, $det(f_\beta) = det(X)^2 N(\lambda) = D(\mathbf{F}_\sharp) N(\lambda)$. ■

Proposition 3.20 *Assume M is of type $O_{\frac{n}{\alpha}}(3^\alpha)$ with $n = 2m+1$ and $\alpha \mid n$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $\alpha = 3$. In this case M has 3 orbits on $\mathfrak{E}_\xi(V)$ and equation (3.1) holds for all M-orbits on $\mathfrak{E}_\xi(V)$ with $r = s$, and* *hence M is in Table 1.1.*

Proof. Let $q = 3^\alpha$, and μ, ν be the generators for $\mathbf{F}_\sharp^*$ and $Gal(\mathbf{F}_\sharp/\mathbf{F})$, respectively. As n is odd, it follows that α is also odd. Write $\alpha = 2\alpha_1 + 1$ and $\dfrac{n}{\alpha} = 2b+1$. Then $m = b\alpha + \alpha_1$, where $n = 2m+1$. Multiplying by a suitable constant to the quadratic form $Q_\sharp$, we can assume that $D(Q_\sharp) = \square$. By Proposition 2.8, there exists a basis $\beta_\sharp = \{w_1, w_2, \cdots, w_{2b+1}\}$ of $(V_\sharp, Q_\sharp)$ such that $f_{\beta_\sharp} = I_{2b+1}$. Define $\phi_\sharp = \phi_{Q_\sharp, \beta_\sharp} = \phi_{\beta_\sharp}(\nu)$. Then $o(\phi_\sharp) = \alpha$. We will show that $\phi_\sharp \in \Omega$. Let $\beta_n = \{\zeta, \zeta^3, \ldots, \zeta^{3^{\alpha-1}}\}$ be a normal basis of $\mathbf{F}_\sharp$ over $\mathbf{F}$, and $\beta_i = \beta_n \otimes w_i$, Since $(\zeta^{3^j} w_i)\phi_\sharp = \zeta^{3^{j+1}} w_i$, we obtain

$$(\phi_\sharp)_{\beta_i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Clearly $det(\phi_\sharp)_{\beta_i} = (-1)^{\alpha-1} = 1$ hence $det(\phi_\sharp) = 1$. Therefore $\phi_\sharp \in S$. As $[S : \Omega] = 2$ and $o(\phi_\sharp) = \alpha$ is odd, $\phi_\sharp \in \Omega$. Let $I = I(V, \mathbf{F}, Q)$, $I_\sharp = I(V_\sharp, \mathbf{F}_\sharp, Q_\sharp)$, and $\Omega_\sharp = \Omega(V_\sharp, \mathbf{F}_\sharp, Q_\sharp)$. Then by (4.3.11) in [29] $M_I = I_\sharp\langle\phi_\sharp\rangle \cong I_\sharp.\mathbb{Z}_\alpha$. Since $\Omega_\sharp$ is perfect, $\Omega_\sharp \leqslant L$, hence $\Omega_\sharp \leqslant M_\Omega N_L(\mathbf{F}_\sharp) \leqslant M_I = I_\sharp\langle\phi_\sharp\rangle$. By Proposition 4.3.17 in [29], $[M_\Omega : \Omega_\sharp] = \alpha$. As $\phi_\sharp \in \Omega \cap I_\sharp\langle\phi_\sharp\rangle$, $\phi_\sharp \in M_\Omega$, and hence $M_\Omega = \Omega_\sharp\langle\phi_\sharp\rangle$. It follows from Lemma 2.3 that $\langle z\rangle\Omega_\sharp = \langle z\rangle I_\sharp$ for any non-singular point $\langle z\rangle$ in $(V_\sharp, \mathbf{F}_\sharp, Q_\sharp)$, so that $\langle z\rangle M_\Omega = \langle z\rangle M_I$, hence we only need to compute parameters for M_Ω in L. We first claim the following:

(1) $Q_\sharp(w\phi_\sharp) = Q_\sharp(w)^\nu$ for any $w \in V_\sharp$.

(2) $\langle z\rangle M_\Omega = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) \in \{\gamma, \gamma^\nu, \ldots, \gamma^{\nu^{\alpha-1}}\}\}$, where $z \in V_\sharp$ with $\gamma = Q_\sharp(z)$. For (1), assume that $w = \sum_{i=1}^{2b+1} \lambda_i w_i \in V_\sharp$. Then $w\phi_\sharp = \sum_{i=1}^{2b+1} (\lambda_i w_i)\phi_\sharp = \sum_{i=1}^{2b+1} \lambda_i^\nu w_i$, hence $Q_\sharp(w\phi_\sharp) = (\sum_{i=1}^{2b+1} -\lambda_i^2)^\nu = Q_\sharp(w)^\nu$. For (2), from (1) we have $Q_\sharp(z\phi_\sharp) = \gamma^\nu$, hence $\{Q_\sharp(z\phi_\sharp^j)\}_{j=1}^{\alpha} = \{\gamma, \gamma^\nu, \cdots, \gamma^{\nu^{\alpha-1}}\}$. Thus $\langle z\rangle M_\Omega = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) \in \{\gamma, \gamma^\nu, \ldots, \gamma^{\nu^{\alpha-1}}\}\}$.

For a non-zero vector $w \in V_\sharp$, consider $span_{\mathbf{F}_\sharp}(w)$ as an α-subspace in V. We consider the case when $\alpha = 3$ and $\alpha > 3$ separately.

(a) **Case** $\alpha > 3$. Let $z \in \{w_1, w_1 + w_2\}$. Then $Q_\sharp(z) = \mp 1$ and $Q(z) = TQ_\sharp(z) = \mp\alpha \neq 0$ so z is non-singular in V. Also as $Q_\sharp(z) = \mp 1$ is fixed under ν, by (2) we have $\langle z\rangle M_\Omega = \langle z\rangle \Omega_\sharp$, and by Lemma 2.11, $|\langle z\rangle M_\Omega| = |\langle z\rangle\Omega_\sharp| = \frac{1}{2}(q^{2b} + \varepsilon.q^b)$, with $\varepsilon = sgn(z^\perp_{V_\sharp})$. We have $\langle z\rangle M_\Omega \cap z^\perp = \langle z\rangle\Omega_\sharp \cap z^\perp = \{v \in V_\sharp \mid Q_\sharp(v) = Q_\sharp(z), Tf_\sharp(v, z) = 0\}$. For $w \in \langle z\rangle M_\Omega \cap z^\perp_V$, write $w = \varphi f_\sharp(z, z)^{-1}z + w_0$, where $w_0 \in z^\perp_{V_\sharp}$, and $T(\varphi) = 0$. Then $f_\sharp(w, z) = \varphi$ and $Q_\sharp(w_0) = Q_\sharp(z)^{-1}(Q_\sharp(z)^2 - \varphi^2)$. As $T(\pm Q_\sharp(z)) = T(\pm 1) \neq 0$, $Q_\sharp(w_0) \neq 0$ for any $\varphi \in \mathbf{F}_\sharp$ with $T(\varphi) = 0$. When $\varphi \in \mathrm{Ker}T$ is fixed, as $dim_{\mathbf{F}_\sharp}(z^\perp_{V_\sharp}) = 2b$ and $sgnz^\perp_{V_\sharp} = \varepsilon$, by Lemma 2.11, there are $q^{2b-1} - \varepsilon q^{b-1}$ vectors w_0 with $Q(w_0) = Q_\sharp(z)^{-1}(1 - \varphi^2) \neq 0$. Also $dim_{\mathbf{F}}(\mathrm{Ker}T) = \alpha - 1$, we conclude that $d_z = |\langle z\rangle M_\Omega \cap z^\perp_V| = \frac{1}{2}3^{\alpha-1}(q^{2b-1} - \varepsilon q^{b-1}) = \frac{1}{6}(q^{2b} - \varepsilon q^b)$. Thus $c_z = \frac{1}{3}(q^{2b} + 2\varepsilon q^b) - 1 = (\varepsilon 3^{b\alpha} - 1)(\varepsilon 3^{b\alpha-1} + 1)$, and $c_z - 2d_z = \varepsilon q^b - 1 = \varepsilon.3^{b\alpha} - 1$. Assume that equation (3.2) holds. Then $c_z - 2d_z = \varepsilon.3^{b\alpha} - 1 = \xi 3^{m-1} - 1$, where $\xi = sgn(z^\perp_V)$. The last equation yields $m = b\alpha + 1$. Recall that $m = b\alpha + \alpha_1$, hence $\alpha_1 = \frac{1}{2}(\alpha - 1) = 1$, this forces $\alpha = 3$, a contradiction. Now suppose that equation (3.3) holds. Then $(3^{m-1} + \xi)(3^m - \xi + \xi(c - 2d)) = 2c$, hence $(3^{m-1} + \xi)(3^m - \varepsilon\xi 3^{b\alpha} - 2\xi) = 2(3^{b\alpha} - \varepsilon)(3^{b\alpha-1} + \varepsilon)$. We will show that $3^{m-1} + \xi > 2(3^{b\alpha} - \varepsilon)$ and $3^m - \varepsilon\xi 3^{b\alpha} - 2\xi > 3^{b\alpha-1} + \varepsilon$, so that after multiplying these two inequalities side by side, we get a contradiction. For the first inequality, we have $3^{m-1} + \xi \geqslant 3^{b\alpha+\alpha_1-1} - 1 \geqslant 3^{\alpha_1-1}3^{b\alpha} - 1 \geqslant 3.3^{b\alpha} - 1$, as $\alpha_1 \geqslant 2$. Now $2(3^{b\alpha} - \varepsilon) \leqslant 2(3^{b\alpha} + 1)$. Thus, it suffices to show that $3.3^{b\alpha} - 1 > 2(3^{b\alpha} + 1)$. This inequality is equivalent to $3^{b\alpha} > 3$. This is true because $b\alpha > 1$. For the second inequality, we have $3^m - \varepsilon\xi 3^{b\alpha} - 2\xi \geqslant 3^{b\alpha+\alpha_1} - 3^{b\alpha} - 2 = (3^{\alpha_1} - 1)3^{b\alpha} - 2 > 2.3^{b\alpha} - 2 = (3^{b\alpha} + 1) + (3^{b\alpha} - 3) > 3^{b\alpha-1} + 1 \geqslant 3^{b\alpha-1} + \varepsilon$, as $3^{b\alpha} - 3 > 0$.

(b) **Case** $\alpha = 3$. Let ω be a root of $x^3 - x + 1$ in $\overline{\mathbf{F}}$. Then $\langle\omega\rangle = \mathbf{F}^*_\sharp$, and $KerT$ has a basis $\{1, \omega\}$ with $T(\omega^2) = -1$. We have $m = 3b + 1, q = 3^3$. Let $x_1 = \omega w_1, x_2 = \omega^2 w_1, x_3 = \omega^4(w_1 + w_2)$ and $y_1 = \omega(w_1 + w_2), y_2 = \omega^2(w_1 + w_2), y_3 = \omega^4 w_1$. For $i = 1, \cdots, 3$, we have

$Q_\sharp(x_i) \neq 0, Q_\sharp(y_i) \neq 0$, and $Q(x_i) = 1, Q(y_i) = -1$ and so x_i, y_i are non-singular in both $V_\sharp$ and V. Also all $x_i's$ ($y_i's$) belong to different $\Omega_\sharp$-orbits but they are in the same Ω-orbits. For each $i = 1, 2$, we have $x_i{}^\perp_{V_\sharp} = \langle w_2, \cdots, w_{2b+1}\rangle$ and $y_i{}^\perp_{V_\sharp} = \langle w_1 - w_2, w_3, \cdots, w_{2b+1}\rangle$, so that $D(x_i{}^\perp_{V_\sharp}) = \square, D(y_i{}^\perp_{V_\sharp}) = \boxtimes$, and hence by Proposition 2.6, $sgn(x_i{}^\perp_{V_\sharp}) = (-)^b$ and $sgn(y_i{}^\perp_{V_\sharp}) = (-)^{b-1}$, where $dim x_i{}^\perp_{V_\sharp} = dim y_i{}^\perp_{V_\sharp} = 2b$. For $i = 3$, as computation above, we have $sgn(x_3{}^\perp_{V_\sharp}) = (-)^{b-1}$ and $sgn(y_3{}^\perp_{V_\sharp}) = (-)^b$. We now determine the type of x_i and y_i in $(V, \mathbf{F}, Q)$. Let $U = span_{\mathbf{F}_\sharp}(w_3) \perp \cdots \perp span_{\mathbf{F}_\sharp}(w_{2b+1}) \leqslant V$ be an $\mathbf{F}$-subspace. We have $x_1{}^\perp_V = \langle w_1, \omega^2 w_1\rangle \perp span_{\mathbf{F}_\sharp}(w_2) \perp U$, $x_2{}^\perp_V = \langle \omega w_1, (\omega^2 - \omega)w_1\rangle \perp span_{\mathbf{F}_\sharp}(w_2) \perp U$, $x_3{}^\perp_V = \langle(\omega + 1)(w_1 + w_2), (\omega^2 - 1)(w_1 + w_2)\rangle \perp span_{\mathbf{F}_\sharp}(w_1 - w_2) \perp U$, and similarly $y_1{}^\perp_V = \langle(w_1 + w_2), \omega^2(w_1 + w_2)\rangle \perp span_{\mathbf{F}_\sharp}(w_1 - w_2) \perp U$, $y_2{}^\perp_V = \langle\omega(w_1 + w_2), (\omega^2 - \omega)(w_1 + w_2)\rangle \perp span_{\mathbf{F}_\sharp}(w_1 - w_2) \perp U$, $y_3{}^\perp_V = \langle(\omega + 1)w_1, (\omega^2 - 1)w_1\rangle \perp span_{\mathbf{F}_\sharp}(w_2) \perp U$. By Lemma 3.19, $D(span_{\mathbf{F}_\sharp}(w_i)) = N(f_\sharp(w_i, w_i)) = N(1) = 1 = \square$ and $D(span_{\mathbf{F}_\sharp}(w_1 - w_2)) = N(f_\sharp(w_1 - w_2, w_1 - w_2)) = N(-1) = -1 = \boxtimes$. Thus $D(x_i{}^\perp_V) = \boxtimes$ and $D(y_i{}^\perp_V) = \square$ for all $i = 1, \cdots, 3$, and so by Proposition 2.7, $sgn(x_i{}^\perp_V) = (-)^{m-1} = (-)^b$ and $sgn(y_i{}^\perp_V) = (-)^m = (-)^{3b+1} = (-)^{b-1}$. Let $z \in \{x_i, y_i\}$ and $\gamma = Q_\sharp(z)$. We have $\langle z\rangle M_\Omega = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) \in \{\gamma, \gamma^3, \gamma^9\}\} = \bigcup_{j=1}^3 \langle z\phi_\sharp^j\rangle\Omega_\sharp$. Observe that in $(V_\sharp, \mathbf{F}_\sharp, Q_\sharp)$ all vectors $z\phi_\sharp^j, j = 1, \cdots, 3$ have the same type, say $\varepsilon = sgn(z^\perp_{V_\sharp})$. It follows from Lemma 2.11 that $1 + c_z + d_z = 3.\frac{1}{2}(q^{2b} + \varepsilon.q^b) = \frac{1}{2}(3^{6b+1} + \varepsilon 3^{3b+1})$. For any $w \in \langle z\rangle M_\Omega \cap z^\perp_V$, $Q_\sharp(w) \in \{\gamma, \gamma^3, \gamma^9\}$ and $T(\varphi) = 0$, with $\varphi = f_\sharp(w, z)$. Write $w = \varphi f_\sharp(z, z)^{-1}z + w_0$, where $w_0 \in z^\perp_{V_\sharp}$. We have $f_\sharp(w, z) = \varphi$ and $Q_\sharp(w_0) = \gamma^{-1}(\gamma Q_\sharp(w) - \varphi^2)$.

Assume that $i = 1, 2$. Let $z \in \{x_i, y_i\}$ and $\gamma = Q_\sharp(z)$. Then $sgn(z^\perp_V) = sgn(z^\perp_{V_\sharp}) = \varepsilon$. We will show that $Q_\sharp(w_0) \neq 0$ for any $\varphi \in \ker T$. By way of contradiction, suppose that $Q_\sharp(w_0) = 0$. Then $\varphi^2 \in \{\gamma^2, \gamma^4, \gamma^{10}\}$, hence $\varphi \in \{\pm\gamma, \pm\gamma^2, \pm\gamma^5\}$ or $\varphi \in \{\pm\omega^2, \pm\omega^4, \pm\omega^8, \pm\omega^{20}, \pm\omega^{10}\}$. As the trace map is non-zero on these values, we get a contradiction. Thus $Q_\sharp(w_0) \neq 0$. By Lemma 2.11, $d_z = 3\frac{1}{2}(q^{2b-1} - \varepsilon.q^{b-1}).3^{\alpha-1} = \frac{1}{2}(q^{2b} - \varepsilon.q^b)$ as $|\ker T| = 3^{\alpha-1} = 3^2, \dim_{\mathbf{F}_\sharp}(z^\perp_{V_\sharp}) = 2b$, and $\varepsilon = sgn(z^\perp_{V_\sharp})$. Then $c_z = q^{2b} + 2\varepsilon.q^b - 1$. Hence

$c_z - 2d_z = \varepsilon.3q^b - 1 = \varepsilon.3^m - 1$. Therefore equation (3.2) holds.

Assume that $z \in \{x_3, y_3\}$. Then $sgn(z_V^{\perp}) = -sgn(z_{V_\sharp}^{\perp}) = -\varepsilon$ and $\gamma = \pm\omega^8$. Also $Q_\sharp(w_0) = \gamma^{-1}(\gamma Q_\sharp(w) - \varphi^2)$. For any $w \in \langle z\rangle M_\Omega$, $Q_\sharp(w) \in \{\gamma, \gamma^3, \gamma^9\}$. If $Q_\sharp(w) = \gamma$ then $Q_\sharp(w_0) \neq 0$ as $T(\varphi) = T(\pm\gamma) \neq 0$. If $Q_\sharp(w) = \gamma^3$ then $Q_\sharp(w_0) = 0$ if and only if $\varphi \in \{\pm\gamma^2\}$, and similarly, if $Q_\sharp(w) = \gamma^9$ then $Q_\sharp(w_0) = 0$ if and only if $\varphi \in \{\pm\gamma^5\}$. By Lemma 2.11, we have $2d_z = 3^2.(q^{2b-1} - \varepsilon q^{b-1}) + 2[2(q^{2b-1} + \varepsilon(q^b - q^{b-1})) + 7(q^{2b-1} - \varepsilon q^{b-1})]$ or $2d_z = 3^{6b} + \varepsilon 3^{3b+1}$, and hence $c_z = 3^{6b} - 1$, so that $c_z - 2d_z = -\varepsilon 3^{3b+1} - 1 = (-\varepsilon)3^m - 1$. Thus equation (3.2) holds.

Let $\xi = (-)^b = sgn(x_{iV}^{\perp})$ and $\eta = sgn(y_{iV}^{\perp}), i = 1, \cdots, 3$. As $|\langle x_1\rangle M_\Omega| + |\langle x_2\rangle M_\Omega| + |\langle x_3\rangle M_\Omega| = \frac{1}{2}(3^{6b+1} + \xi 3^{3b+1}) + \frac{1}{2}(3^{6b+1} + \xi 3^{3b+1}) + \frac{1}{2}(3^{6b+1} - \xi 3^{3b+1}) = \frac{1}{2}3^{3b+1}(3^{3b+1} + \xi) = \frac{1}{2}3^m(3^m + \xi) = |\mathfrak{E}_\xi(V)|$, M_Ω has exactly three orbits on $\mathfrak{E}_\xi(V)$. Thus equation (3.2) holds for all points in $\mathfrak{E}_\xi(V)$. Similarly M_Ω has three orbits on $\mathfrak{E}_\eta(V)$, and so equation (3.2) holds for all points in $\mathfrak{E}_\eta(V)$. ■

The tensor product subgroups $\mathcal{C}_4$

Let V_i be vector spaces over $\mathbf{F}_q$ of dimension n_i, $i = 1, \cdots, t$. Let $V = V_1 \otimes \cdots \otimes V_t$. For $g_i \in GL(V_i)$, the element $g_1 \otimes \cdots \otimes g_t \in GL(V)$ acts on V as follows:

$$(v_1 \otimes \cdots \otimes v_t)(g_1 \otimes \cdots \otimes g_t) = v_1 g_1 \otimes \cdots \otimes v_t g_t (v_i \in V_i),$$

and extend linearly. Now suppose that q is odd, and that $\mathbf{f}_i$ is a non-degenerate bilinear form on V_i, so that $(V_i, \mathbf{F}, \mathbf{f}_i)$ is either symplectic or orthogonal geometry. We next define the bilinear form $\mathbf{f} = \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_t$ on $V_1 \otimes \cdots \otimes V_t$ by $\mathbf{f}(v_1 \otimes \cdots \otimes v_t, w_1 \otimes \cdots \otimes w_t) = \prod_{i=1}^{t} \mathbf{f}_i(v_i, w_i)$ and extend linearly. We write

$$(V, \mathbf{f}) = (V_1 \otimes \cdots \otimes V_t, \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_t)$$

for such a structure and call *tensor decomposition* and denote by $\mathcal{D}$. The members of $\mathcal{C}_4(\mathit{\Gamma})$ is the stabilizer of tensor decomposition $\mathcal{D}$ such that

(a) $(V, \mathbf{f}) = (V_1 \otimes V_2, \mathbf{f}_1 \otimes \mathbf{f}_2)$,

(b) $(V_1, \mathbf{f}_1)$ is not similar to $(V_2, \mathbf{f}_2)$,

(c) $\mathbf{f}_i$ are symmetric, $(n_1, \varepsilon_1) \neq (n_2, \varepsilon_2)$, where $n_i = dimV_i \geqslant 3$, $\varepsilon_i = sgnV_i$, or $\mathbf{f}_i$ are symplectic, $dimV_1 < dimV_2$.

Proposition 3.21 *Assume M is of type $O_{n_1}(3) \otimes O_{n_2}(3)$, with $n_1 < n_2$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. Let $v = v_1 \otimes v_2 \in V$, where $v_i \in V_i$ are non-singular vectors. Then $\langle v \rangle$ is a non-singular point. By (4.4.14)[29], $M_I = I_1 \otimes I_2$. We have $\Omega_1 \times \Omega_2 \trianglelefteq M_\Omega \leqslant I_1 \otimes I_2 = M_I$. As Ω_i act transitively on $\mathfrak{E}(V_i)$, it follows that $\langle v \rangle M_I = \langle v \rangle M_\Omega = \langle v_1 \otimes v_2 \rangle (\Omega_1 \times \Omega_2) = \langle v_1 \rangle \Omega_1 \otimes \langle v_2 \rangle \Omega_2$. Therefore $\langle v \rangle M_\Omega = \langle v \rangle M$ since $M_\Omega \leqslant M \leqslant M_I$. Write $n_1 = 2a+1, n_2 = 2b+1$. Then $|\langle v \rangle M_\Omega| \leqslant |\langle v_1 \rangle \Omega_1|.|\langle v_2 \rangle \Omega_2| \leqslant \frac{1}{2}.3^a(3^a+1).3^b(3^b+1) < 3^{2(a+b)}$ (as $3^a + 3^b + 1 < 3^{a+b}$ and $b > a \geqslant 1$). Since $dimV = 2m+1 = (2a+1)(2b+1)$, $m = 2ab + a + b$. Then $m - 1 - 2(a+b) = a(b-2) + b(a-1) + a - 1 \geqslant 0$ so that $3^{m-1} \geqslant 3^{2(a+b)} > |\langle v \rangle M_\Omega|$. This violates (3.4) so that equation (3.1) cannot hold. ■

The tensor product subgroups $\mathcal{C}_7$

Let V_1 be an α-dimensional vector space over $\mathbf{F} = \mathbf{F}_q$, and assume that $\mathbf{f}_1$ is either 0, a non-degenerate bilinear form, or a non-degenerate unitary form. For $i = 1, \cdots, b$, let $(V_i, \mathbf{f}_i)$ be a classical geometry which is similar to $(V_1, \mathbf{f}_1)$. For each i, denote by η_i the similarity from $(V_1, \mathbf{f}_1)$ to $(V_i, \mathbf{f}_i)$ satisfying $\mathbf{f}_i(v\eta_i, w\eta_i) = \lambda_i \mathbf{f}_1(v, w)$ for all $v, w \in V_1$, where $\lambda_i \in \mathbf{F}^*$ is independent of v and w. Thus we obtain a tensor decomposition $\mathcal{D}$ given by $(V, \kappa) = (V_1, \mathbf{f}_1) \otimes \cdots \otimes (V_b, \mathbf{f}_b)$, where $V = V_1 \otimes \cdots \otimes V_b$ and $\kappa = Q(\mathbf{f}_1, \cdots, \mathbf{f}_b)$ if q is even and each $\mathbf{f}_i$ is symplectic and $\kappa = \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_b$ otherwise. Let $X_i = X(V_i, \mathbf{f}_i)$ for

$X \in \{\Omega, S, I, \Lambda, \Xi, A\}$. Define $\Xi_{\mathcal{D}} = \Xi_{(\mathcal{D})} S_b$. The members of $\mathcal{C}_7(\Xi)$ are the groups $\Xi_{\mathcal{D}}$ with $b \geqslant 2$ described as above.

Proposition 3.22 *Assume M is of type $O_\alpha(3) \wr S_b$ with $\alpha \geqslant 5$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. We have $\Omega_\alpha(3) \wr S_b \trianglelefteq M_\Omega \leqslant O_\alpha(3) \wr S_b$. Let $v_1 = v \otimes v \otimes \cdots \otimes v$ and $v_2 = v_1 + w \otimes w \otimes \cdots \otimes w \in V$, where $v \neq w$ belong to some orthogonal basis of V_1. As S_b fixes v_i, $\langle v_i \rangle M_\Omega = \langle v_i \rangle (\Pi_{i=1}^b \Omega_\alpha(3))$. Also $\langle v_i \rangle M_\Omega = \langle v_i \rangle M_I = \langle v_i \rangle M$. For $i = 1, 2$, the stabilizers of $\langle v_i \rangle$ in $\Pi_{i=1}^b \Omega_\alpha(3)$ contain a subgroup which is isomorphic to $\Pi_{i=1}^b \Omega_{\alpha-2}(3)$. Thus $|\langle v_i \rangle M_\Omega| = |\langle v_i \rangle (\Pi_{i=1}^b \Omega_\alpha(3))| \leqslant [\Omega_\alpha(3) : \Omega_{\alpha-2}(3)]^b < 3^{(4a-3)b}$, where $a = \frac{\alpha-1}{2} \geqslant 2$. Now $m - 1 = \frac{1}{2}((2a+1)^b - 3)$. Consider the following function in variable $x \in [2, +\infty)$, where $b \geqslant 2$,

$$f(x) := \frac{1}{2}((2x+1)^b - 3) - (4x-3)b.$$

We have $f'(x) = b(2x+1)^{b-1} - 4b \geqslant b(2x+1) - 4b \geqslant b > 0$. Hence $f(x) \geqslant f(2) = \frac{1}{2} g(b)$, where $g(b) = 5^b - 10b - 3$ and $b \geqslant 2$. By induction on $b \geqslant 2$, $g(b) > 0$. Thus $3^{m-1} > 3^{b(4a-3)} > |\langle v_i \rangle M_\Omega|$. This contradicts (3.4). Thus equation (3.1) cannot hold. ∎

3.4.3 Permutation characters of maximal subgroups in $\mathcal{S}$

In this section, we consider the maximal subgroup $M \in \mathcal{S}(\overline{G})$. By Definition 2.12, M is an almost simple group and the socle S of M is a non-abelian simple group. Then the full covering group $\widehat{S}$ of S acts absolutely irreducible on V, the natural module for G and preserves a non-degenerate quadratic form on V.

Recall the construction of the fully deleted permutation module for alternating groups as in the discussion before Proposition 2.23. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be a standard basis for $\mathbf{F}_p^n$, and let $w_0 = \varepsilon_1 + \cdots + \varepsilon_n \in \mathbf{F}_p^n$. Put $U = w_0^\perp, W = \mathbf{F}_p w_0$, and $V = U/(U \cap W)$. Define $e_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1$. Then $\{e_i\}_{i=1}^{n-1}$ is a basis for V if p does not divide n, and

$\{e_i + U \cap W\}_{i=1}^{n-2}$ is a basis for V if $p|n$. Let Q be the quadratic form on V induced from the quadratic form associated to the natural bilinear form on $\mathbf{F}_p^n$. Then $(V, \mathbf{F}_p, Q)$ is a classical orthogonal geometry and $A_n \leqslant \Omega(V)$. To simplify the notation, we always write e_i instead of $e_i + U \cap W$.

Lemma 3.23 *Assume M is almost simple of type A_n with $n \geqslant 10$ and V is the fully deleted permutation module for A_n in characteristic 3. Let $v = \varepsilon_1 - \varepsilon_2, w = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \in V$. Then*

(1) $|\langle v\rangle M| = \frac{1}{2}n(n-1), d_v = \frac{1}{2}(n-2)(n-3),$ *and* $c_v = 2n - 4$;

(2) $|\langle w\rangle M| = \frac{1}{8}n(n-1)(n-2)(n-3), c_w = 2n^3 - 25n^2 + 111n - 172$ *and*

$d_w = 2 + 4(n-4)^2 + \frac{1}{8}(n-4)(n-5)(n-6)(n-7)$.

Proof. As A_n acts transitively on V, $xM = xA_n = xS_n$ for any $x \in V$. Since $v = \varepsilon_1 - \varepsilon_2$, it is clear that if $g \in S_n$ and $(\varepsilon_1 - \varepsilon_2)g = \varepsilon_1 - \varepsilon_2$, then g must fix indices 1 and 2. Thus $(S_n)_v \simeq S_{n-2}$. Similarly, $(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)g = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$ implies that g must fix the partitions $\{1, 2\}, \{3, 4\}$. Thus $g \in S_2 \times S_2 \times S_{n-4}$. Therefore $|M : M_{\langle v\rangle}| = \frac{1}{2}[S_n : S_{n-2}] = \frac{1}{2}n(n-1)$, and $|M : M_{\langle w\rangle}| = \frac{1}{2}[S_n : S_2 \times S_2 \times S_{n-4}] = \frac{1}{8}n(n-1)(n-2)(n-3)$.

(i) Parameters for v. We have $\langle u\rangle \in \langle v\rangle M \cap v^\perp$ if and only if $u = \varepsilon_i - \varepsilon_j \notin \mathbf{F}_p v, i \neq j$ and $(\varepsilon_i - \varepsilon_j, \varepsilon_1 - \varepsilon_2) = 0$. This happens only if $\{i, j\} \cap \{1, 2\} = \emptyset$, or $i, j \in \{3, 4, \ldots, n\}$. There are $\binom{n-2}{2}$ such points $\langle u\rangle$. Thus $d_v = |\langle v\rangle M \cap v^\perp| = \frac{1}{2}(n-2)(n-3),$ and $c_v = 2n-4$.

(ii) Parameters for w.

We will show that $d = |\langle w\rangle M \cap w^\perp| = 2 + 4(n-4)^2 + \frac{1}{8}(n-4)(n-5)(n-6)(n-7)$ and $c = 2n^3 - 25n^2 + 111n - 172$. For any $\langle u\rangle \in \langle w\rangle M \cap w^\perp$, $u = \varepsilon_i + \varepsilon_j - \varepsilon_r - \varepsilon_s$, where $|\{i, j, r, s\}| = 4$, and $(u, w) = 0$. Denote by *supp*(u) the set of non-zero indices of ε_i appearing in u. We consider the cases:

(1) *supp*$(u) \cap$ *supp*$(w) = \emptyset$. It follows that *supp*$(u) \in \{5, 6, \ldots, n\}$. Hence there are $(n-4)(n-5)(n-6)(n-7)/8$ points.

(2) $|$*supp*$(u) \cap$ *supp*$(w)| = 1$. There are no such u, since $(u, w) \neq 0$.

(3) $|supp(u) \cap supp(w)| = 2$. Suppose that $i, j \in \{1, 2, 3, 4\}$. Then either $u = \varepsilon_i - \varepsilon_j + \varepsilon_r - \varepsilon_s$, where $\varepsilon_i - \varepsilon_j \in \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4\}$, or $u = \varepsilon_i + \varepsilon_j - \varepsilon_r - \varepsilon_s$, where $\varepsilon_i + \varepsilon_j \in \{\varepsilon_1 + \varepsilon_3, \varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_3, \varepsilon_2 + \varepsilon_4\}$; and $r, s \in \{5, \ldots, n\}$. There are $2(n-4)(n-5)$ and $4\binom{n-4}{2}$ points respectively. Thus there are $4(n-4)(n-5)$ points in this case.

(4) $|supp(u) \cap supp(w)| = 3$. Suppose that $i, j, r \in \{1, 2, 3, 4\}$. Then $u = \pm\varepsilon_i \pm \varepsilon_j \pm \varepsilon_r \pm \varepsilon_s$, where $s \in \{5, \ldots, n\}$, $\varepsilon_i, \varepsilon_j, \varepsilon_r$ with their signs appearing exactly as in w, and sign of ε_s is chosen so that there are 2 minuses and 2 pluses. There are $\binom{4}{3}(n-4) = 4(n-4)$.

(5) $|supp(u) \cap supp(w)| = 4$. There are just 2 points in this case: $\{\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4, \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4\}$. Therefore $d_w = \frac{1}{8}(n-4)(n-5)(n-6)(n-7) + 2(n-4)(n-5) + 4(n-4)(n-5) + 2 = \frac{1}{8}(n-4)(n-5)(n-6)(n-7) + 2(n-4)^2 + 2$, and $c_w = \frac{1}{8}n(n-1)(n-2)(n-3) - d - 1$. ■

Proposition 3.24 *Assume M is almost simple of type A_n, with $n \geqslant 10$, and V is the fully deleted permutation module for A_n in characteristic $p = 3$. Further assume that $n - 1 - \varepsilon_3(n) = 2m + 1$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. Let $v = \varepsilon_1 - \varepsilon_2, w = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \in V$. Then $Q(v) = 1$, and $Q(w) = -1$. Hence v, w are non-singular vectors in V. We see that $n - 1 - \varepsilon_3(n)$ is odd if and only if $n = 6k+2, n = 6k+3$ or $n = 6k+4$. By Lemma 3.23(1) $|\langle v \rangle M| = \frac{1}{2}n(n-1)$. If $n \geqslant 13$, then $\frac{1}{2}(n-5) > \log_3(\frac{1}{2}n(n-1))$. Since $m - 1 = \frac{1}{2}(n - 1 - \varepsilon_3(n)) - 1 \geqslant \frac{1}{2}(n-5)$, as $\varepsilon_3(n) \leqslant 2$. It follows that $1 + c + d = \frac{1}{2}n(n-1) < 3^{m-1}$. This violates (3.4) and so equation (3.1) cannot hold. Hence, we only need to consider $10 \leqslant n \leqslant 12$. Since $n = 6k+2, 6k+3$ or $6k+4$, it follows that $n = 10$. Then $n - 1 - \varepsilon_3(n) = 9, d_v = 28, c_v = 16, m = 4$. If equation (3.1) holds, then either $c_v - 2d_v = \xi 3^4 - 1 = 81\xi - 1 = -40$, or $(27\xi + 1)(81\xi - 41) = 32$. These equations clearly cannot hold with $\xi = \pm 1$. By Lemma 3.23(2), $|\langle w \rangle M| = n(n-1)(n-2)(n-3)/8$. If $n \geqslant 23$, then $(3^m + 1)/2 > n(n-1)(n-2)(n-3)/8$ so that equation (3.1) cannot hold by (3.4). Thus we can assume that $10 \leqslant n \leqslant 22$. Then $n \in \{10, 14, 15, 16, 20, 21, 22\}$.

(*a*) $n = 10$. Then $n - \varepsilon_3(n) = 9, m = 4$, $d = 191, c = 438$ and $c - 2d = 56$.

Table 3.2: Low degree representations of small Alternating groups.

A_n	A_6	A_7	A_7	A_8	A_8	A_8	A_9	A_9
$\dim D^\lambda$	9	13	15	7	13	21	21	7
λ	$(4,2)$	$(5,2)$	$(5,1^2)$	$(7,1)$	$(6,2)$	$(6,1^2)$	$(7,1^2)$	$(8,1)$
$m(\lambda)$	$(2^2,1^2)$	$(3,2,1^2)$	$(3,2^2)$	$(4,3,1)$	$(3^2,1^2)$	$(3^2,2)$	$(4,3,2)$	$(4^2,1)$

(b) $n = 14$. Then $n - \varepsilon_3(n) = 13, m = 6$, $d = 1032, c = 1970$ and $c - 2d = -94$.

(c) $n = 15$. Then $n - \varepsilon_3(n) = 13, m = 6$, $d = 1476, c = 2618$ and $c - 2d = -334$.

(d) $n = 16$. Then $n - \varepsilon_3(n) = 15, m = 7$, $d = 2063, c = 3396$ and $c - 2d = -730$.

(e) $n = 20$. Then $n - \varepsilon_3(n) = 19, m = 9$, $d = 6486, c = 8048$ and $c - 2d = -4924$.

(f) $n = 21$. Then $n - \varepsilon_3(n) = 19, m = 9$, $d = 8298, c = 9656$ and $c - 2d = -6940$.

(g) $n = 22$. Then $n - \varepsilon_3(n) = 21, m = 10$, $d = 10478, c = 11466$ and $c - 2d = -9490$.

We can check that equation (3.1) cannot hold in any of these cases. ∎

Proposition 3.25 *Assume M is almost simple of type A_n with $n \geqslant 12$, and V is not the fully deleted permutation module for A_n in characteristic $p = 3$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. As $n \geqslant 12$, by Lemma 2.32, we have $dim(V) = 2m + 1 \geqslant \frac{1}{2}(n^2 - 5n + 2)$ so that $m \geqslant \frac{1}{4}(n^2 - 5n)$. However when $n \geqslant 12$, $3^{m-1} > 3^{\frac{n^2-5n-4}{4}} > n! = |Aut(A_n)|$. Thus equation (3.1) cannot hold which in view of (3.4). ∎

Proposition 3.26 *Assume M is almost simple of type $S = A_n$ with $5 \leqslant n \leqslant 11$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $n = 9$ and $V \cong D^{(8,1)}$, in which case M has at most 2 orbits on $\mathfrak{E}_\xi(V)$ so that $1_P^G \nleqslant 1_M^G$ by Corollary 3.7 and hence $(L, S) = (\Omega_7(3), A_9)$ is in Table 1.2.*

Proof. Using the information on the p-modular representations of alternating groups and their covering groups in [26], we only need to consider the cases given in Table 3.2.

(i) Let $\lambda = (8, 1)$. Then $m(\lambda) = (4^2, 1) \neq \lambda$. By Theorem 2.31, $D^\lambda \downarrow_{A_9}$ is irreducible. Thus

$A_9 \leqslant S_9 \leqslant \Omega_7(3)$, and there is two classes of S_9 in $\Omega_7(3)$. As $8-1+1+1=9 \equiv 0(\mathrm{mod}\ 3)$, λ is a JS-partition (see Definition 2.22), and hence by Theorem 2.29, $D^\lambda \downarrow_{S_8} = D^{\lambda(1)} = D^{(7,1)}$. Then since $(7,1) \neq m(7,1) = (4,3,1)$, we have: $A_8 \leqslant S_8 \leqslant S_9$. In this case, D^λ is the fully deleted permutation module for S_9 over $\mathbf{F_3}$. Then $n-1-\varepsilon_3(n) = 7$, and $m = 3$. Let $v = \varepsilon_1 - \varepsilon_2, w = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \in V$. There are only two orbits of type $\rho_V(v)$, with representatives $v = \varepsilon_1 - \varepsilon_2$ and $\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5 = \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7-\varepsilon_8$, and one orbit of type $\rho_V(w)$. Thus equation (3.1) holds for both types of points.

(ii) If $\lambda = (7,1)$ or $\lambda = (4,3,1)$, then $A_8 < S_8 < S_9 < \Omega_7(3)$, since $D^{(8,1)} \downarrow_{S_8} = D^{(7,1)}$ and $D^{(7,1)} \downarrow_{A_8}$ is irreducible.

(iii) If $\lambda = (6,1^2)$ or $(3^2,2)$, then $A_8 < S_8 < S_9 < \Omega_{21}(3)$, since $D^{(7,1^2)} \downarrow_{S_8} = D^{(6,1^2)}$ and $D^{(6,1^2)} \downarrow_{A_8}$ is irreducible.

(iv) If $\lambda = (5,2)$ or $(3,2,1^2)$, then $A_7 < S_7 < S_8 < \Omega_{13}(3)$.

(v) Now, if $\lambda = (n-2,1^2)$, where $n = 7$ or 9, then $D^\lambda = \wedge^2(D^{(n-1,1)})$. As $D^{(n-1,1)}$ is the fully deleted permutation module for S_n, we can apply the construction above for fully deleted module. Let $v = e_1 \wedge e_3 = (\varepsilon_1 - \varepsilon_2) \wedge (\varepsilon_3 - \varepsilon_4)$, and $w = (e_1 - e_2) \wedge (e_3 - e_4) = (\varepsilon_1+\varepsilon_2+\varepsilon_3) \wedge (\varepsilon_3+\varepsilon_4+\varepsilon_5)$. Then v, w are non-singular points of different types in D^λ. We have $|vS_n| = \frac{1}{2}\frac{n!}{2.(n-4)!}$ and $|wS_n| = \frac{1}{2}\frac{n!}{2.(n-5)!}$. We then get contradictions by using (3.4).

(vi) If $\lambda = (4,2)$ then $dimD^\lambda = 9, m = 4$, and we have an embedding $A_6 \leqslant S_6 \leqslant \Omega_9(3)$. However, as $A_6 \cong L_2(9) < A_{10} < \Omega_9(3)$, hence $M = N_G(A_6)$ is not maximal in G.

(vii) If $\lambda = (6,2)$, then $dimV = 13, m = 6$. We have $A_8 \leqslant S_8 \leqslant \Omega_{13}(3)$. Using GAP, S_8 has two points w_1, w_2 of different types with the same orbit sizes 315. We have $(c,d) = (194,120)$ or $(c,d) = (212,102)$. We see that equations (3.2) and (3.3) cannot hold. ■

Groups of Lie type: Representations in cross-characteristic

Proposition 3.27 *Assume M is almost simple of type S, where S is a finite simple Chevalley group in cross-characteristic. There is an M-orbit on $\mathfrak{C}_\xi(V)$ such that equation (3.1) does not hold unless $(L,S) = (\Omega_7(3), PSp_6(2))$, in which case M has only two orbits*

Table 3.3: Small groups in cross-characteristic.

S	$(3^{\frac{e(S)-1}{2}}+1)/2 \leqslant \lvert Aut(S)\rvert$
$L_2(q)$	$L_2(q), 3 \leqslant q \leqslant 68$
$L_n(q), n \geqslant 3$	$L_3(2), L_4(2), L_5(2), L_3(4)$
$PSp_{2n}(q), n \geqslant 2$	$S_4(5), S_4(7), S_6(5)$
	$S_4(2), S_6(2), S_8(2), S_4(4)$
$U_n(q), n \geqslant 3$	$U_n(2), 3 \leqslant n \leqslant 7, U_3(4), U_3(5)$
$P\Omega^+_{2n}(q), n \geqslant 4$	$\Omega^+_8(2)$
$P\Omega^-_{2n}(q), n \geqslant 4$	$\Omega^-_8(2)$
$\Omega_{2n+1}(q), n \geqslant 3, q$ odd	
$E_6(q)$	
$E_7(q)$	
$E_8(q)$	
$F_4(q)$	$F_4(2)$
${}^2E_6(q)$	
$G_2(q)$	
${}^3D_4(q)$	${}^3D_4(2)$
${}^2F_4(q)$	${}^2F_4(2)'$
$Sz(q)$	$Sz(8)$
${}^2G_2(q)$	

on $\mathfrak{E}_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7, and so (L,S) is in Table 1.2.

Proof. Suppose equation (3.1) holds for some $r \in \{s,t\}$ and for some M-orbit $\langle x\rangle M$ with $\langle x\rangle \in \mathfrak{E}_\xi(V)$. Then $|\langle x\rangle M| \geqslant \frac{1}{2}(3^m+1)$, by (3.4). On the other hand, using the lower bounds for degrees of cross-characteristic representations of finite Chevalley groups given in Table 2.6, we have $2m+1 \geqslant e(S)$, so that $(3^m+1)/2 \geqslant (3^{(e(S)-1)/2})/2$. Moreover $|\langle x\rangle M| \leqslant |M| \leqslant |Aut(S)|$. It follows that $(3^{\frac{e(S)-1}{2}}+1)/2 \leqslant |Aut(S)|$. By this condition, we get a finite list as in Table 3.3. Next we shorten the list by using information on cross-characteristic representations of small groups in the preliminaries together with [26] and [13]. Further V is an absolutely irreducible $\mathbf{F}_3\widehat{S}$-module which is self-dual with $ind(V) = +1$, and $dimV$ is odd.

(*i*) Case $S = L_2(q), 2 \leqslant q \leqslant 68$. As $L_2(2), L_2(3)$ are not simple, $L_2(4) \cong L_2(5) \cong A_5$, and q is a prime power, we can assume that $7 \leqslant q \leqslant 67$.

Case $q \equiv 1 \pmod 4$. By Table 2.7(a) and the fact that $dimV$ is odd, we have $dimV \in \{(q+1)/2, q\}$. Using (3.4) again, we only need to consider the following cases:

(1) $q \in \{13, 17, 29, 37, 41, 49\}$ and $dimV = (q+1)/2$;

(2) $q = 13, 17$ and $dimV = q$.

If $dimV = (q+1)/2$, then q must be a square in $\mathbf{F}_3$, so that $q \equiv 1 \pmod 3$ and hence $q = 13, 37, 49$. If $dimV = q$, then $3 \nmid (q+1)$, so that $q = 13$.

If $(S, dimV) = (L_2(13), 7)$ then $L_2(13) < G_2(3) < \Omega_7(3)$ by [9] so M is not maximal in G.

If $(S, dimV) = (L_2(13), 13)$ then $L_2(13) < \Omega_{13}(3)$. Let w be a non-singular eigenvector of an element of order 13. Then $|\langle w\rangle S| = 14$ and hence $|\langle w\rangle M| \leqslant |Out(S)|.|\langle w\rangle S| = 2.14 < 3^{m-1} = 3^5$ so that equation (3.1) cannot hold. For other type of point, if $M = S$ then there exists a point $\langle u\rangle$ with $|\langle u\rangle S| = 1092$ and $(c,d) = (734, 357)$. If $M = S.2$ then there exists a point $\langle v\rangle$ with $|\langle v\rangle M| = 2184$ and $(c,d) = (1469, 714)$. We check that equation (3.1) cannot hold in any of these cases.

If $(S, dimV) = (L_2(37), 19)$ then $m = 9$ and the eigenvector w of an element of order 37 is non-singular and $|\langle w\rangle S| = 38$, $|Out(S)| = 2$ and hence $|\langle w\rangle M| \leqslant 38.2 < 3^{m-1}$, so that equation (3.1) cannot hold for this point. For other type of point, there exists a point $\langle u\rangle$ with $|\langle u\rangle S| = |S| = 25308$ and $(c,d) = (16919, 8388)$. We see that equation (3.1) cannot hold.

If $(S, dimV) = (L_2(49), 25)$ then $m = 12$ and there exist two non-singular vectors of different type $u_+, u_- \in V$ which are eigenvectors of an element of order 7 in S such that $|\langle u_\xi\rangle S| \leqslant 8400, \xi = \pm$. As $|Out(S)| = 4$ and $|\langle u_\xi\rangle M| \leqslant |Out(S)|.|\langle u_\xi\rangle S| \leqslant 4.8400 < 3^{m-1}$. In view of (3.4), equation (3.1) cannot hold.

Case $q \equiv 3 \pmod 4$. As in previous case, by Table 2.7(b), we have $dimV = q$ and $3 \nmid q+1$. Using (3.4) again, we get $q \in \{7, 19\}$.

If $(S, dimV) = (L_2(7), 7)$ then $L_2(7) < \Omega_7(3)$ but $\Omega_7(3)$ has no maximal subgroup with socle $L_2(7)$ by [9].

If $(S, dimV) = (L_2(19), 19)$ then $m = 9$ and there exist two non-singular vectors of different type $u_+, u_- \in V$ which are eigenvectors of an element of order 5 in S such that $|\langle u_\xi \rangle S| \leqslant 342, \xi = \pm$. As $|Out(S)| = 2$ and $|\langle u_\xi \rangle M| \leqslant |Out(S)|.|\langle u_\xi \rangle S| \leqslant 2.342 < 3^{m-1}$. In view of (3.4) equation (3.1) cannot hold.

Case $q \equiv 0 \pmod 2$. As in previous case, by Table 2.7(c), we have $dimV \in \{q-1, q+1\}$. Using (3.4) again, we have $q = 8, 16$. By [26], the only possibility is $q = 8$ and $dimV = 7$. However by [9], $L_2(8) \leqslant G_2(3) \leqslant \Omega_7(3)$.

(ii) Case $L_n(q), (n, q) \in \{(3, 2), (4, 2), (5, 2), (3, 4)\}$. As $L_3(2) \cong L_2(7)$, and $L_4(2) \cong A_8$, which have been done above, we can exclude these groups.

If $S = L_3(4)$, then $Out(S) \cong 2 \times S_3 \cong D_{12}$. By [26], $dimV \in \{15, 19, 45, 63\}$. By using (3.4), we only need to consider the representations of degrees 15 and 19.

Assume first that $dimV = 15$. Then $m = 7$. If $M = L_3(4)$ then there exists two non-singular vectors $u_i, i = 1, 2$ with $|\langle u_i \rangle L_3(4)| = 2016, (c_1, d_1) = (1250, 765)$, and $(c_2, d_2) = (1350, 660)$. Similarly, if $M = L_3(4).2_1$ then $(c_1, d_1) = (2600, 1431), (c_2, d_2) = (1250, 765)$, if $M = L_3(4).2_2$ or $L_3(4).2_3$ then $(c_1, d_1) = (2720, 1311), (c_2, d_2) = (2810, 1221)$, if $M = L_3(4).2^2$ then $(c_1, d_1) = (1250, 765), (c_2, d_2) = (5327, 2736)$. We check that equation (3.1) cannot hold.

Assume that $dimV = 19$. Then $m = 9$. If $M = L_3(4)$ then there exist two non-singular vectors of different type $u_i, i = 1, 2$ which are the eigenvectors of elements of order $7, 5$, respectively and $(c_1, d_1) = (707, 252), (c_2, d_2) = (1190, 825)$. For the remaining extensions of $L_3(4)$, there exist two non-singular vectors of different type $u_i, i = 1, 2$, which are the eigenvectors of an element of order 5 such that the parameters $(c_i, d_i), i = 1, 2$, are as follows: If $M = L_3(4).2_2$ or $M = L_3(4).2_3$ then $(c_1, d_1) = (1190, 825), (c_2, d_2) = (1100, 915)$. If $M = L_3(4).2_3$ then $(c_1, d_1) = (2561, 1470), (c_2, d_2) = (1100, 915)$. If $M = L_3(4).S_3$ or $L_3(4).D_{12}$ then $(c_1, d_1) = (3842, 2205), (c_2, d_2) = (4220, 1827)$. If $M = L_3(4).3$

then $(c_1, d_1) = (3932, 2115), (c_2, d_2) = (4220, 1827)$. We can check that equation (3.1) cannot hold in any of these cases.

Finally if $S = L_5(2)$, then $dimV \geqslant 155$. But $(3^m + 1)/2 \geqslant (3^{77} + 1)/2 > |Aut(L_5(2))|$ so that equation (3.1) cannot hold.

(iii) Case $S \in \{S_4(5), S_4(7), S_6(5)\}$. If $S \cong S_4(5)$ or $S_6(5)$ then the smallest odd degree non-trivial irreducible representations of S has degree 13, and 63, respectively. However since the smallest field of definitions of these representations are quadratic extensions of $\mathbf{F}_3$, (cf. [19]), L can not embed in $\Omega_{13}(3)$ and $\Omega_{63}(3)$. By Theorem 2.1, in [16], if Φ is a representation of S which is not the smallest representation, then $dim\Phi \geqslant (q^n - 1)(q^n - q)/(2(q+1))$, which are 40 and 1240, respectively. But then inequality (3.4) cannot hold. If $S = S_4(7)$, then the smallest non-trivial representation in characteristic 3 of S is a Weil representation of degree 25. However, the Frobenius -Schur indicator is $\circ$, (cf. [19]), which means that $S_4(7)$ fixes no quadratic form. Thus $S_4(7)$ cannot embed in $\Omega_{25}(3)$. If Φ is a non-trivial representation of $S_4(7)$ which is not the smallest representation of S, then $dim(\Phi) \geqslant 126$, but this again violates (3.4). Thus equation (3.1) cannot hold.

(iv) Case $S \in \{S_4(2), S_6(2), S_8(2), S_4(4)\}$. By the isomorphism $S_4(2) \cong S_6$, it follows that $A_6 \cong S_4(2)'$. Thus we can exclude this case. If $S = S_4(4)$ then $dimV \geqslant 51$ and $(3^m + 1)/2 \geqslant (3^{25} + 1)/2 > |Aut(S_4(4))|$ so equation (3.1) cannot hold. For $S_6(2)$, and $S_8(2)$, we need to consider the following cases $(S_6(2), 7), (S_6(2), 21), (S_6(2), 27)$ $(S_8(2), 35)$.

If $S = S_6(2)$ and $dim(V) = 7$, then by [9], $S_6(2)$ is a maximal subgroup of $\Omega_7(3)$ and it has only two orbits on $\mathfrak{E}(V)$ so that equation (3.1) holds for both types of points.

If $(S, dimV) = (S_6(2), 21)$ then $m = 10$, $Out(S) = 1$ and $S_6(2) \leqslant \Omega_{21}(3)$. There exist two non-singular vectors of different type $u_i, i = 1, 2$ which are the eigenvectors of elements of order $5, 12$, respectively and $(c_1, d_1) = (212, 165), (c_2, d_2) = (2132, 1647)$.

If $(S, dimV) = (S_6(2), 27)$ then $m = 13$ and $S_6(2) \leqslant \Omega_{27}(3)$. There exist two non-singular vectors of different type $u_i, i = 1, 2$ which are the eigenvectors of an element of order 5 and $(c_1, d_1) = (96968, 48183), (c_2, d_2) = (47912, 24663)$.

If $(S, dimV) = (S_8(2), 35)$ then $m = 17, Out(S_8(2)) = 1$ and $S_8(2) \leqslant \Omega_{35}(3)$. There exist two non-singular vectors of different type $u_i, i = 1, 2$ which are the eigenvectors of an element of order 5 and $(c_1, d_1) = (119, 0), (c_2, d_2) = (256094, 129465)$ and so equation (3.1) cannot hold.

(v) Case $S \in \{U_n(2), 3 \leqslant n \leqslant 7, U_3(4), U_3(5)\}$. As $U_3(2) \cong 3^2.Q_8$ and $U_4(2) \cong S_4(3)$, we can rule out these cases. If $S = U_5(2)$, then by [26], the smallest odd degree non-trivial 3-modular representation of S has degree 55. Thus $(3^m + 1)/2 \geqslant (3^{27} + 1)/2 > |Aut(S)|$. If $S = U_3(4)$, then $dimV \in \{13, 39, 75\}$. However if $dimV \geqslant 39$ then (3.4) cannot hold, and if $dimV = 13$, then by [19], $U_3(4)$ fixes no quadratic form. If $S = U_7(2)$, then by [19], we have $dimV > 250$ and so (3.4) cannot hold. For the remaining cases, using [26], we need to consider the following cases: $(U_3(5), 21)$ and $(U_6(2), 21)$.

If $(M, dimV) = (U_3(5), 21)$ then we have $m = 10$, $Out(S) = S_3$ and $M \leqslant \Omega_{21}(3)$. For each extension M of S, there exist two non-singular vectors of different type with parameters $(c_i, d_i), i = 1, 2$, as follow: if $M = U_3(5)$ then $(c_1, d_1) = (7033, 3466), (c_2, d_2) = (27145, 14854)$; if $M = U_3(5).2$ then $(c_1, d_1) = (55437, 28562), (c_2, d_2) = (55371, 28628)$.

If $(S, dimV) = (U_6(2), 21)$ then we have $m = 10$, $M \leqslant \Omega_{21}(3)$ and $Out(S) = S_3$. For each extension M of S, there exist two non-singular vectors of different type with parameters $(c_i, d_i), i = 1, 2$, as follow: if $M = U_6(2)$ then $(c_1, d_1) = (7033, 3466), (c_2, d_2) = (27145, 14854)$; if $M = U_6(2).2$ then $(c_1, d_1) = (55437, 28562), (c_2, d_2) = (55371, 28628)$. We can check that equation (3.1) cannot hold in any of these cases.

(vi) Case $S = O_8^+(2)$. By [26], the smallest odd degree non-trivial 3-modular representation of S has degree 35, and the second smallest odd degree one has degree 147. Using (3.4) again, we only need to consider the 3-modular representation of $O_8^+(2)$ of degree 35. We have $m = 17$ and there are two non-singular points $v_i, i = 1, 2$, of different type with $|\langle v_1 \rangle S| = |120$ and $|\langle v_2 \rangle S = |90720$. Then $|\langle v_i \rangle M| \leqslant |Out(S)||\langle v_i \rangle S| < 3^{m-1}$ and so equation (3.1) cannot hold.

(vii) Case $S = O_8^-(2)$. By [26], $dimV \geqslant 203$. Then inequality (3.4) cannot hold.

$(viii)$ Case $S = F_4(2)$. By [19], $dimV > 255$. Then inequality (3.4) cannot hold.

(ix) Case $S = G_2(4)$. By [26], $dimV = 2m+1 \geqslant 649$. Clearly, $3^{m-1} > 4^{15} \geqslant |Aut(S)|$.

(x) Case $S = Sz(8)$. By [26], $dimV = 35$. Then inequality (3.4) cannot hold.

(xi) Case $S = {}^3D_4(2)$. By [26], either $dimV = 25$ or $dimV \geqslant 351$. If the latter case hold then $m \geqslant 174$ and clearly $3^{m-1} \geqslant 3^{174} > 3.2^{29} \geqslant |Aut(S)|$. When $dimV = 25$, the group S is not maximal in $\Omega_{25}(3)$ as ${}^3D_4(2) \leqslant F_4(3) \leqslant \Omega_{25}(3)$.(see[35]).

(xii) Case $S = {}^2F_4(2)'$. By [26], $2m+1 \geqslant 77$. Then $3^{m-1} \geqslant 3^{37} > 2^{27} > |Aut(S)|$. ■

Groups of Lie type: Representations in defining characteristic

Let k be an algebraically closed field of characteristic p. Let $\mathfrak{G}$ be a simply connected, simple algebraic group over k. Fix a maximal torus T and a Borel subgroup B containing T. Let U be the unipotent radical of B. Then $B = UT$. Let Φ be the root system of $\mathfrak{G}$, select a system of positive roots Φ^+ from Φ, with corresponding fundamental roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. Let $\{\lambda_1, \ldots, \lambda_\ell\}$ be the fundamental dominant weights and X^+ be the set of dominant weights. Let $\mathfrak{L}$ be the simple Lie algebra over $\mathbb{C}$ of the same type as $\mathfrak{G}$. For each dominant weight $\lambda \in X^+$, there exists an irreducible $\mathfrak{L}$-module $V(\lambda)$ of highest weight λ, and a maximal vector v^+ (unique up to scalar multiplication). Let $\mathcal{U} = \mathcal{U}(\mathfrak{L})$ be the universal enveloping algebra of $\mathfrak{L}$, and $\mathcal{U}_{\mathbb{Z}}$ be the Kostant $\mathbb{Z}$-form of $\mathcal{U}$ (see [21], 26.3). Now, $\mathcal{U}_{\mathbb{Z}}v^+$ is the minimal admissible lattice in $V(\lambda)$, and $\mathcal{U}_{\mathbb{Z}}v^+ \otimes_{\mathbb{Z}} k$ is a $k\mathfrak{G}$-module of highest weight λ, also denoted by $V(\lambda)$, and called a *Weyl module* for $\mathfrak{G}$ ([21], 27.3). The Weyl module $V(\lambda)$ has a unique maximal submodule $J(\lambda)$ and $L(\lambda) = V(\lambda)/J(\lambda)$ is an irreducible $k\mathfrak{G}$-module of highest weight λ.

Assume that $\widehat{S}$ is simply connected of type A_ℓ or ${}^2A_\ell$ over $\mathbf{F_q}$, where $q = p^f$, and $\mathfrak{G}$ be the corresponding simply connected, simple algebraic group over k, such that $\widehat{S} = \mathfrak{G}_\sigma$ for some suitable Frobenius map σ. Let $N = N(\widehat{S})$ be the natural module for $\widehat{S}$. We collect here some information about $L(\lambda)$ for some special dominant weights λ.

(1) Let $0 < c < p$, and $\lambda = c\lambda_1$ or $\lambda = c\lambda_\ell$. Then $L(\lambda)$ has all weight spaces of dimension 1. $L(\lambda)$ is isomorphic to the space of homogeneous polynomials of degree c, that is, $L(\lambda) \cong S^c(N)$. In particular, $\dim L(\lambda) = \dfrac{(\ell+c)!}{\ell!c!}$ ([42], 1.14).

(2) If $\ell > 1$, then $L(\lambda_i) \cong \bigwedge^i N$ and $\dim L(\lambda_i) = \binom{\ell+1}{i}$ (see [7]).

(3) Let $\lambda = n_1\lambda_1 + n_2\lambda_2 + \dots n_\ell\lambda_\ell$ be a dominant weight. Then $L(\lambda)$ preserves a non-degenerate bilinear form if and only if $n_1 = n_\ell, n_2 = n_{\ell-1}, \dots$. Thus if ℓ is even then $L(\lambda_i)$ leaves invariant no non-degenerate bilinear form, and if ℓ is odd then $L(\lambda_i), i \neq \frac{\ell+1}{2}$ does not preserve any such form. Let $\lambda = \lambda_{(\ell+1)/2}$. If $\ell \equiv -1 \pmod 4$ then $L(\lambda)$ fixes a symmetric bilinear form and it fixes an alternating bilinear form if $\ell \equiv 1 \pmod 4$. (see [5] Chapter VIII §13 Table 1, p. 217).

The following constructions for adjoint modules of groups of type A_ℓ and ${}^2A_\ell$ are taken from [36], pp.491 − 492.

(4) We construct the irreducible module $L(\lambda_1 + \lambda_\ell)$ as follows: Let $V := V_1/(V_1 \cap V_2)$, where $V_1 = \{A \in M_{\ell+1}(q) \mid Tr(A) = 0\}, V_2 = \{aI_{\ell+1} \mid a \in \mathbf{F}_q\}$. Let $\widehat{S}$ act on V_1 by conjugation. Then V is an irreducible $\widehat{S}$-module of dimension $\ell^2 + 2\ell - \varepsilon_p(\ell+1)$. The bilinear form on V_1 is defined as follows: for any $A, B \in V_1$, $(A, B) = Tr(AB)$. We can check that $\widehat{S}$ preserves this bilinear form. Also, V has a basis consisting of $E_{i,j}, 1 \leqslant i < j \leqslant \ell+1, E_{ii} - E_{i+1,i+1}, i = 1, \cdots, \ell - \varepsilon_p(\ell+1)$.

(5) Let $\widehat{S} = SU_n(q)$ and $\lambda = \lambda_1 + \lambda_\ell$, where $n = \ell + 1$. Let $V_2 = \{aI_n \mid a \in \mathbf{F}_{q^2}\}$, $V_1 = \{A \in M_n(q^2) \mid Tr(A) = 0, A = \overline{A}^t\}$, and set $V := V_1/(V_1 \cap V_2)$, where the map $A \mapsto \overline{A}$ is the map that raises each entry to its q^{th}-power. Let $\widehat{S}$ act on V_1 as in (4). The bilinear form on V_1 is also defined as in (4). We can check that $\widehat{S}$ preserves this bilinear form and $L(\lambda_1 + \lambda_\ell) \cong V$. Moreover fix a generator μ of $\mathbf{F}^*_{q^2}$, V has a basis consisting of $E_{i,j} + E_{j,i}, \mu E_{i,j} + \overline{\mu} E_{j,i} 1 \leqslant i < j \leqslant \ell+1, E_{ii} - E_{i+1,i+1}, i = 1, \cdots, \ell - \varepsilon_p(\ell+1)$.

Proposition 3.28 *Assume M is almost simple of type S, where $\widehat{S}$ is simply connected of* *type A_ℓ or ${}^2A_\ell$ over $\mathbf{F}_{3^f}$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s,t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$. We consider the case $f=1$ and $f>1$ separately.

Case $f=1$. We can assume that $\ell \geqslant 2$. By Theorems 2.35 and 2.36, there exists a 3-restricted dominant weight $\lambda \in X_3$ such that $V \cong L(\lambda)$. As $|Aut(S)| = |Aut(L^{\varepsilon}_{\ell+1}(3))| \leqslant 3^{(\ell+1)^2}$, where $\varepsilon = \pm$. It follows from (3.4) that $3^{m-1} < 3^{(\ell+1)^2}$. Hence $m < (\ell+1)^2+1$ and $dimV < 2(\ell+1)^2+3$. We need to look for all dominant weights $\lambda \in X_3$ such that $L(\lambda)$ is self-dual, has dimension less than $2(\ell+1)^2+3$ and of odd degree. If $\ell \geqslant 18$, then $\dfrac{\ell^3}{8} \geqslant 2(\ell+1)^2+3$, and so by Theorem 5.1 in [38], λ is one of the following 3-restricted dominant weights $\{\lambda_1, \lambda_\ell, \lambda_2, \lambda_{\ell-1}, 2\lambda_1, 2\lambda_\ell, \lambda_1+\lambda_\ell\}$. Since $L(\lambda)$ is self-dual, the only possibility for λ is $\lambda_1+\lambda_\ell$. If $\ell < 18$, then by Theorem 4.4, Appendix A_6 through A_{21} in [38], either $\lambda = 2\lambda_2$ when $\ell=3$ or $\lambda = \lambda_1+\lambda_\ell$ for $2 \leqslant \ell \leqslant 17$.

Suppose that $\ell \geqslant 4$. We have $dimL(\lambda_1+\lambda_\ell) = \ell^2+2\ell-\varepsilon_p(\ell+1)$. As $dimL(\lambda_1+\lambda_\ell)$ is odd, it follows that $\ell = 6b_1+1, 6b_1+2$ or $6b_1+3$ for $b_1 \geqslant 1$. Consequently $\ell \geqslant 7$. As constructed above, $L(\lambda_1+\lambda_\ell) \cong V := V_1/(V_1 \cap V_2)$. Let U be the subgroup of $\widehat{S}$ consisting of all matrices of the form $diag(I_2, A)$, where $A \in SL^{\varepsilon}_{\ell-1}(3)$. Then $U \cong SL^{\varepsilon}_{\ell-1}(3)$. For $\xi = \pm 1$, let $x_\xi = E_{1,2}+\xi E_{2,1}+V_1\cap V_2$, when $\varepsilon = +$, and $x_+ = E_{1,2}+E_{2,1}+V_1\cap V_2$, $x_- = \mu E_{1,2}+\overline{\mu}E_{2,1}+V_1\cap V_2$ when $\varepsilon = -$. Then $x_\xi \in V$ and $Q(x_\xi) \neq 0$. It follows that $\langle x_\xi \rangle$ is a non-singular point in V, of plus or minus type depending on ξ and ℓ. As $V_1 \cap V_2$ is fixed under natural action of U, and clearly, U centralizes x_ξ, it follows that $U \leqslant S_{\langle x_\xi \rangle}$, the stabilizer of $\langle x_\xi \rangle$ in S. We have $1+c+d \leqslant |Aut(S):U| = [Aut(L^{\varepsilon}_{\ell+1}(3)) : SL^{\varepsilon}_{\ell-1}(3)] \leqslant 2.3^{2\ell-1}(3^\ell+1)(3^{\ell+1}+1) < 3^{4\ell+2}$.

As $2m+1 = \ell^2+2\ell-\varepsilon_3(\ell+1) \geqslant \ell^2+2\ell-1$, $m-1 \geqslant \frac{1}{2}(\ell^2+2\ell-4)$. We have $\frac{1}{2}(\ell^2+2\ell-4)-(4\ell+2) = \frac{1}{2}(\ell(\ell-6)-8)$. If $\ell \geqslant 8$ then $\ell(\ell-6)-8 > 0$, hence $m-1 > 4\ell+2$. If $\ell=7$ then $2m+1 = \ell^2+2\ell = 63$, or $m=31$ and $m-1 = 30 = 4\ell+2$. Thus $m-1 \geqslant 4\ell+2$ for any $\ell \geqslant 4$. Hence $3^{m-1} \geqslant 3^{4\ell+2} > 1+c+d$. This contradicts to inequality (3.4). Therefore, equation (3.1) cannot hold in this case.

We are left with the cases $\ell = 2, \lambda = \lambda_1 + \lambda_2$, $\ell = 3, \lambda = \lambda_1 + \lambda_3$, and $\ell = 3, \lambda = 2\lambda_2$. If the first case holds then $\widehat{S} = SL_3^{\varepsilon}(3)$, and $dimL(\lambda_1 + \lambda_2) = 7$. However, by [9], $\Omega_7(3)$ has no maximal subgroup with socle $L_3^{\varepsilon}(3)$.

Assume that $(S, L) = (L_4(3), \Omega_{15}(3))$. For each extension of S, using [13], we can find two non-singular points of different type with parameters (c, d) as follow: if $M = L_4(3)$ then $(c, d) = (42524, 20655)$, $(1160, 945)$, if $M = L_4(3).2$ then $(c, d) = (311768, 154791)$, $(505196, 252963)$. Assume that $(S, L) = (U_4(3), \Omega_{15}(3))$. As in case $L_4(3)$, for each extension M of $U_4(3)$, we can find two non-singular points of different type with the parameters (c, d) as follow: If $M = U_4(3)$ or $U_4(3).2$ then $(c, d) = (435212, 217971)$, $(2780, 1755)$; if $M = U_4(3).2^2$ then $(c, d) = (217970, 108621)$, $(435212, 217971)$; if $M = U_4(3).4$ or $U_4(3).D_8$ then $(c, d) = (217970, 108621)$.

If $(S, L) = (L_4(3), \Omega_{19}(3))$ and $M = L_4(3), L_4(3).2$ then there exists two non-singular points of different types with $(c, d) = (2600, 1611), (1070, 1035)$.

Assume $(S, L) = (U_4(3), \Omega_{19}(3))$. For each extension M of $U_4(3)$, we can find two non-singular points of different type with the parameters (c, d) as follow: if $M = U_4(3), U_4(3).2$ then $(c, d) = (2690, 1845), (217700, 108891)$; if $M = U_4(3).4$ then $(c, d) = (435752, 217431)$, $(217700, 108891)$; if $M = U_4(3).2^2, U_4(3).D_8$ then $(c, d) = (435752, 217431), (2420, 2115)$. We can check that equation (3.1) cannot hold in any of these cases.

Case $f > 1$. First consider case $\ell = 1$. As $SL_2(q) \cong SU_2(q)$, we can assume that $\varepsilon = +$. If $f = 2$, then $S = SL_2(9)$. Then $\overline{S} = L_2(9) \cong A_6$. Thus, we can assume that $f \geqslant 3$. If λ is any 3-restricted dominant weight then $\lambda = c\lambda_1$, where $0 \leqslant c \leqslant 2$, $dimL(c\lambda_1) = c + 1$ and $L(c\lambda_1)$ is self-dual. By Proposition 2.41, $dimV = (dim\Psi)^f$, for some irreducible $k\widehat{S}$-module Ψ. As $dimV = 2m + 1$ is odd, $dim\Psi$ is odd and hence $dim\Psi \geqslant 3 = dimL(2\lambda_1)$. It follows that $2m + 1 \geqslant 3^f$ and hence $m - 1 \geqslant (3^f - 3)/2$. As $|Aut(L_2(3^f))| = f \cdot 3^f(3^{2f} - 1) < 3^{4f}$, it follows from (3.4) that $3^f - 3)/2 \leqslant 4f$, with

$f \geqslant 3$. However by induction on $f \geqslant 3$, this is not true. Thus equation (3.1) cannot hold.

Consider case $\ell \geqslant 2$. It is shown in case $f = 1$ that if $\lambda \in X_3$ such that $L(\lambda)$ is self-dual and has smallest odd degree then $\lambda = \lambda_1 + \lambda_\ell$. By Propositions 2.41 and 2.39 again, $2m+1 = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ of odd degree. Hence $dim\Psi \geqslant dimL(\lambda_1+\lambda_\ell)$. It follows that $2m+1 \geqslant (\ell^2+2\ell-\varepsilon_3(\ell+1))^f$. We will show that $3^{m-1} > |Aut(L^{\varepsilon}_{\ell+1}(3^f))|$. Then (3.4) cannot hold. As $|Aut(L^{\varepsilon}_{\ell+1}(3^f))| < 3^{f(\ell+1)^2}$ and $m-1 \geqslant ((\ell^2+2\ell-\varepsilon_3(\ell+1))^f-3)/2 \geqslant ((\ell^2+2\ell-1)^f-3)/2$, it suffices to show that $((\ell^2+2\ell-1)^f-3)/2 > f(\ell+1)^2$. This is true by induction. The proof is now completed.■

Let $\widehat{S}$ be a simply connected group of type B_ℓ over $\mathbf{F}_q$, where $q = p^f$, and $\mathfrak{G}$ be the corresponding simply connected, simple algebraic group over k, such that $S = \mathfrak{G}_\sigma$ for some suitable Frobenius map σ. Let N be the natural module for $\widehat{S}$ with the standard basis $\beta = \{e_1,\dots,e_\ell,x,f_1,\dots,f_\ell\}$. Multiplying some suitable constant to the symmetric bilinear form, we can assume that the representing matrix of the symmetric bilinear form on N has the form

$$B = \begin{pmatrix} 0 & 0 & I_\ell \\ 0 & 1 & 0 \\ I_\ell & 0 & 0 \end{pmatrix}.$$

Let T be the set of all matrices of the form $diag(d,1,d^{-1})$, where $d = diag(t_1,\dots,t_\ell) \in GL_\ell(k)$. As $T \cong (k^*)^\ell$, T is a maximal torus of $\widehat{S}$. For $i = 1\dots\ell$, define $\gamma_i : T \to k^*$, by $\gamma_i(diag(t_1,\dots,t_\ell,1,t_1^{-1},\dots,t_\ell^{-1})) = t_i$. Then $\{\gamma_i\}_{i=1}^{\ell}$ form an orthonormal basis for E. Also define $\alpha_{\ell+1-i} = \gamma_{\ell+1-i} - \gamma_{\ell-i}$, for $i = 1,\dots,\ell-1$, and $\alpha_1 = \gamma_1$. Then $\{\alpha_1,\dots,\alpha_\ell$ is a fundamental root system of type B_ℓ, and the corresponding $\mathbb{Z}$-basis of the fundamental dominant weights is $\{\lambda_1,\dots,\lambda_\ell\}$, defined as following: $\lambda_1 = \frac{1}{2}(\gamma_1+\dots+\gamma_\ell)$, and $\lambda_{\ell+1-i} = \gamma_\ell+\gamma_{\ell-1}+\dots+\gamma_{\ell+1-i}$, for $i = 1\dots,\ell-1$.

Proposition 3.29 *Assume M is almost simple of type S, where $\widehat{S}$ is simply connected of type B_ℓ over $\mathbf{F}_{3^f}$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold* *so that M is not in Tables 1.1-1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$.

Case $f = 1$**.** First we claim that if λ is a 3-restricted dominant weight such that $dimL(\lambda)$ is odd and greater than $dimN$ then λ must be one of the following weights:

(i) $\lambda = \lambda_{\ell-1}, \ell \geqslant 3, \ell$ odd, and $dimL(\lambda) = 2\ell^2 + \ell$;

(ii) $\lambda = 2\lambda_\ell, \ell = 6k+3, 6k+4$ or $6k+5$, for some non-negative integer k, and $dimL(\lambda) = 2\ell^2 + 3\ell, 2\ell^2 + 3\ell - 1, 2\ell^2 + 3\ell$, respectively;

(iii) $\ell = 3, \lambda = 2\lambda_1$, and $dimL(\lambda) = 35$.

From (3.4), we have $3^{m-1} \leqslant |Aut(S)| = |Aut(\Omega_{2\ell+1}(3))| = 3^{\ell^2} \prod_{i=1}^{\ell}(3^{2i} - 1) \leqslant 3^{2\ell^2+\ell}$. Hence $dimL(\lambda) = 2m + 1 \leqslant 4\ell^2 + 2\ell + 3$. Notice that if $\ell \geqslant 5$ then $\ell^3 - (4\ell^2 + 2\ell + 3) \geqslant 5\ell^2 - 4\ell^2 - 2\ell - 3 = (\ell - 1)^2 - 4 > 0$, and so $\ell^3 > 4\ell^2 + 2\ell + 3$. If $\ell > 11$, then $dimL(\lambda) \leqslant 4\ell^2 + 2\ell + 3 < \ell^3$, and hence by Theorem 5.1 in [38], λ is either $\lambda_{\ell-1}$ or $2\lambda_\ell$. For $2 \leqslant \ell \leqslant 11$, by Theorem 4.4 in [38] and the upper bound for dimension of $L(\lambda)$ above, again, λ is one of the weights above or $\ell = 3$, $\lambda = 2\lambda_1$, and $dimL(2\lambda_1) = 35$. So case (iii) holds. It remains to get the restriction on ℓ in cases (i) and (ii). From the reference above, we also have $dimL(\lambda_{\ell-1}) = \ell(2\ell+1)$ and $dimL(2\lambda_\ell) = 2\ell^2 + 3\ell - \varepsilon_3(2\ell+1)$. Now case (i) holds as $dimL(\lambda_{\ell-1})$ is odd if and only if ℓ is odd. Suppose that $3 \mid 2\ell+1$. then as $2\ell+1$ is odd, $2\ell+1 = 3(2t+1)$, hence $\ell = 3t+1$. Since $dimL(2\lambda_\ell) = 2\ell^2+3\ell-1 = \ell(2\ell+3)-1$ is odd, it follows that $\ell = 3t+1$ is even. Thus $t = 2k+1$ and $\ell = 6k+4$. With the same argument, we can see that if $3 \nmid 2\ell+1$, then $\ell = 6k+3$ or $6k+4$.

Let Q be the non-degenerate quadratic form associated with the non-degenerate symmetric bilinear form on N. Then for $v \in N, (v, v) = 2Q(v)$. On the tensor product $N \otimes N$, we can define a non-degenerate symmetric bilinear form induced from the form on N as follows: for $u_i, w_i \in N, i = 1, 2$, $(u_1 \otimes u_2, w_1 \otimes w_2) = (u_1, w_1)(u_2, w_2)$, and extend linearly on $N \otimes N$. Recall that if u is a non-singular vector in N, then the reflection $r_u : N \to N$ defined by $vr_u = v - \dfrac{(v, u)}{Q(u)}u$, for any $v \in N$. Let $\widehat{S}$ act on $N \otimes N$ by

$(u \otimes v)g = (ug \otimes vg)$. Then $((u_1 \otimes u_2)r_u, (w_1 \otimes w_2)r_u) = ((u_1 \otimes u_2, w_1 \otimes w_2)$ for any non-singular vector u. Thus, $\wedge^2 N$ and $S^2(N)$ leave invariant symmetric bilinear forms induced from the one on $N \otimes N$. We have $L(\lambda_{\ell-1}) \cong \wedge^2(N)$ and $L(2\lambda_\ell) \cong w^\perp/(w^\perp \cap \langle w \rangle)$, where $w = \sum_{i=1}^{\ell}(e_i \otimes f_i + f_i \otimes e_i) + x \otimes x \in S^2(N)$.

We now consider case (i). As $L(\lambda_{\ell-1}) \cong \wedge^2 N$, $dim L(\lambda_{\ell-1}) = \ell(2\ell+1)$ and $L(\lambda_{\ell-1})$ has a basis consisting of $e_i \wedge e_j, f_i \wedge f_j, 1 \leqslant i < j \leqslant \ell$, $e_i \wedge f_j, 1 \leqslant i, j \leqslant \ell$ and $e_i \wedge x, x \wedge f_i, 1 \leqslant i \leqslant \ell$. Also denote by Q the associated quadratic form on $N \otimes N$. Then for $\xi \in \{\pm 1\}$, let $v = e_1 \wedge x + \xi x \wedge f_1 = (e_1 - \xi f_1) \wedge x$. Since $Q(e_1 \wedge x) = 0 = Q(x \wedge f_1)$, we have $Q(v) = (e_1 \wedge x, x \wedge f_1) = \xi$. Hence v is a non-singular point. Let N_1 be the subspace of N generated by $\{e_1 - \xi f_1, x\}$. As N_1 is non-degenerate, $N = N_1 \perp N_1^\perp$. Denote by H the centralizer of N_1 in $\Omega(N) \cong \Omega_{2\ell+1}(3)$. It follows that $H \cong \Omega_{2\ell-1}(3)$, and H fixes v. By (3.4) we have $3^{m-1} \leqslant 1 + c + d \leqslant |Aut(\Omega_{2\ell+1}(3)) : \Omega_{2\ell-1}(3)| = 2 \cdot 3^{2\ell-1}(3^{2\ell} - 1) \leqslant 3^{4\ell}$. Hence $m - 1 < 4\ell$, so that $2\ell^2 + \ell = 2m + 1 < 8\ell + 3$. As ℓ is odd and $\ell > 1$, the above inequality holds only when $\ell = 3$. In this case, we have $\Omega_7(3) \leqslant \Omega_{27}(3)$. Using GAP, there are two non-singular points of different type $\langle x_i \rangle, i = 1, 2$, with $(c_i, d_i) = (13040, 9072), (26324, 17901)$, we see that equation (3.1) cannot hold in this case. In case (ii), let $v = e_1 \otimes e_1 + \xi f_1 \otimes f_1 + \langle w \rangle \cap w^\perp$, where $\xi \in \{\pm 1\}$. Then $Q(v) = \xi$, hence v is non-singular in $L(2\lambda_\ell)$. Let $N_1 = \langle e_1, f_1 \rangle$. Then N_1 is a non-degenerate subspace of N. As in case (i), let H be the centralizer of N_1 in $\Omega(N)$, as $H \cong \Omega_{2\ell-1}(3)$, we have $3^{m-1} \leqslant 1 + c + d \leqslant |Aut(\Omega_{2\ell+1}(3)) : \Omega_{2\ell-1}(3)| < 3^{4\ell}$, hence $2m + 1 < 8\ell + 3$. As $dim L(2\lambda_\ell) = 2\ell^2 + 3\ell - \varepsilon_3(2\ell+1)$, it follows that $2\ell^2 + 3\ell - 1 \leqslant 2\ell^2 + 3\ell - \varepsilon_3(2\ell+1) < 8\ell + 3$. If $\ell \geqslant 4$ then $2\ell^2 + 3\ell - 1 \geqslant 2 \cdot 4\ell + 3\ell - 1 = 8\ell + (3\ell - 1) > 8\ell + 11 > 8\ell + 3$, and if $\ell = 3$, then $2\ell^2 + 3\ell - \varepsilon_3(2\ell+1) = 27 = 8\ell + 3$. Thus since $\ell \geqslant 3$ in this case, (3.4) cannot hold. Finally $\ell = 3$ and $\lambda = 2\lambda_1$. In this case, we have $\Omega_7(3) \leqslant \Omega_{27}(3)$. Using GAP, there are two non-singular points of different type $\langle x_i \rangle, i = 1, 2$, with $(c_i, d_i) = (13850, 8262), (26324, 17901)$, we see that equation (3.1) cannot hold in this case.

Case $f \geqslant 2$. By Propositions 2.41 and 2.39, $dimV = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ. As $dimV$ is odd, so is $dim\Psi$. Firstly, suppose that $f \geqslant 3$. Since $dim\Psi$ is at least $2\ell+1$, it follows that $2m+1 \geqslant (2\ell+1)^f$. By (3.4), we have $3^{m-1} \leqslant |Aut(\Omega_{2\ell+1}(3^f))| < f \cdot 3^{f(2\ell^2+\ell)} \leqslant 3^{f(2\ell^2+\ell+1)}$. As $2m+1 \geqslant (2\ell+1)^f$, we have $\frac{1}{2}((2\ell+1)^f-3) < f\cdot(2\ell^2+\ell+1)$. Clearing fraction, we get $(2\ell+1)^f-3-f\cdot(4\ell^2+2\ell+2) < 0$. By induction, this inequality cannot happen.

Secondly, suppose that $f = 2$ and $dim\Psi > 2\ell+1$. It follows from case $f = 1$ that $dim\Psi \geqslant 2\ell^2+\ell$. Arguing as above, we have $(\ell(2\ell+1))^2-3 < 2\cdot 2\cdot(2\ell^2+\ell+1) = 2(4\ell^2+2\ell+2)$. However, as $\ell \geqslant 2$, $\ell^f(2\ell+1)^f - 3 - 2(4\ell+2\ell+2) \geqslant 2^2(2\ell+1)^2 - 2(2\ell+1)^2+4\ell-5 \geqslant 2(2\ell+1)^2+3 > 0$. Hence $\ell^f(2\ell+1)^f-3 > 2(4\ell^2+2\ell+2)$, a contradiction.

Finally, suppose that $f = 2$ and $dim\Psi = 2\ell+1$. In this case, we can assume that $\Psi \cong L(\lambda_\ell) \cong N$, and so $V = N \otimes N^{(1)}$, where N is the natural module for $\Omega_{2\ell+1}(3^2)$, and $N^{(1)}$ denote the module received from the twist action of $\widehat{S}$ on N. For any element $v \in N$, denote by $v^{(1)}$ the corresponding element in $N^{(1)}$. Notice that if p is odd then for any $a, b \in \mathbf{F}_p^f$, $a+b = 0$ or 1 if and only if $a^p+b^p = 0$ or 1, correspondingly. This holds because $a^p+b^p = (a+b)^p$. Fix a standard basis $\beta = \{e_1, \cdots, e_\ell, x, f_\ell, \cdots, f_1\}$ of S. Let $u = (\varepsilon_1+\xi f_1) \in N$. Then for $g \in S$, in the basis β, we write $g = (a_{i,j})$. Assume that $ug = g$. Then $a_{11}+a_{2\ell+1,1} = 1 = a_{1,2\ell+1}+a_{2\ell+1,2\ell+1}$ and $a_{1i}+a_{2\ell+1,i} = 0, 1 < i < 2\ell+1$. Hence, by the notice above, we have $a_{11}^p+a_{2\ell+1,1}^p = 1 = a_{1,2\ell+1}^p+a_{2\ell+1,2\ell+1}^p$ and $a_{1i}^p+a_{2\ell+1,i}^p = 0, 1 < i < 2\ell+1$. Therefore, $u^{(1)}g = ug^\nu = u(a_{ij}^p) = u = u^{(1)}$. This means that if $g \in \widehat{S}$ fixes u then g also fixes $u^{(1)}$. Let $v = u \otimes u^{(1)} \in N \otimes N^{(1)}$. Let H be the stabilizer of u in $\widehat{S}$. Then $H \cong \Omega_{2\ell}^\varepsilon(3^2)$ with $\varepsilon = \pm 1$, and $H \leqslant \widehat{S}_v$. Hence by (3.4), $3^{m-1} \leqslant |Aut(\Omega_{2\ell+1}(3^2)) : \Omega_{2\ell}^\varepsilon(3^2)| \leqslant |Aut(\Omega_{2\ell+1}(3^2)) : \Omega_{2\ell}^+(3^2)| \leqslant 3^{4\ell+2}$. Since $2m+1 = (2\ell+1)^2$, it follows that $(2\ell+1)^2 < 8\ell+7$. As $\ell \geqslant 2$, $(2\ell+1)^2-8\ell-7 = 4\ell^2+4\ell+1-8\ell-7 = 4\ell(\ell-1)-6 > 0$. This final contradiction finishes the proof. ■

Let $\widehat{S}$ be a simply connected group of type C_ℓ over $\mathbf{F}_q$, where $q = p^f$, and $\mathfrak{G}$ be the corresponding simply connected, simple algebraic group over k, such that $\widehat{S} = \mathfrak{G}_\sigma$ for some suitable Frobenius map σ. Let N be the natural module for $\widehat{S}$ with the standard basis $\beta = \{e_1, \ldots, e_\ell, f_1, \ldots, f_\ell\}$. The representing matrix of the non-degenerate symplectic form on N has the form $B = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$. From the isomorphisms $Sp_2(q) \cong SL_2(q)$, and $Sp_4(q) \cong \Omega_5(q)$ for q odd, we can assume that $\ell \geqslant 3$.

Proposition 3.30 *Assume M is almost simple of type S, where $\widehat{S}$ is simply connected of type C_ℓ over $\mathbf{F}_{3^f}$, with $\ell \geqslant 3$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $(\ell, \lambda, dimV) = (3, \lambda_2, 13)$ or $(L, S, \lambda) = (\Omega_{41}(3), PSp_8(3), \lambda_1)$. If the first case holds then M has at most two orbits on $\mathfrak{E}_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7 and hence $(L, S) = (\Omega_{13}(3), PSp_6(3))$ is in Table 1.2. The last case is in Table 1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$.

Case $f = 1$. Let $\lambda \in X_3$ be a 3-restricted dominant weight such that $L(\lambda) \cong V$. We first get an upper bound for $dimV$. As $|Aut(PSp_{2\ell}(3))| = f \cdot 3^{\ell^2} \prod_{i=1}^{\ell}(3^{2i} - 1) \leqslant 3^{2\ell^2+\ell}$, by (3.4), $3^{m-1} \leqslant 3^{2\ell^2+\ell}$, and hence $2m + 1 \leqslant 4\ell^2 + 2\ell + 3$. If $\ell \geqslant 5$ then $4\ell^2 + 2\ell + 3 < \ell^3$, and so $dimV < \ell^3$. If $\ell > 11$, then $dimL(\lambda) < \ell^3$, and hence by Theorem 5.1, in [38], λ is either $\lambda_{\ell-1}$ or $2\lambda_\ell$. For $2 \leqslant \ell \leqslant 11$, by Theorem 4.4 in [38] and the upper bound for dimension of $L(\lambda)$ above, again, λ is one of the weights above or $\ell = 4$, $\lambda = \lambda_1$, and $dimL(\lambda_1) = 41$. We have $dimL(2\lambda_\ell) = \ell(2\ell + 1)$, $L(2\lambda_\ell) \cong S^2(N)$, and $dimL(\lambda_{\ell-1}) = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell), L(\lambda_{\ell-1}) \cong w^\perp/(\langle w \rangle \cap w^\perp)$, where $w = e_1 \wedge f_1 + \cdots + e_\ell \wedge f_\ell$. In these cases, $\widehat{S}$ leaves invariant a quadratic form Q induced from the symplectic form on N. In case $\lambda = 2\lambda_\ell$, let $v = e_1 \otimes e_1 + \xi f_1 \otimes f_1 \in L(2\lambda_\ell)$. Since $dimL(2\lambda_\ell) = \ell(2\ell + 1)$ is odd, ℓ must be odd.

If $\ell = 3$ then $dimL(2\lambda_\ell) = 21$ and $PSp_6(3) \leqslant \Omega_{21}(3)$. Using [13], there are two non-singular points of different type $\langle x_i \rangle, i = 1, 2$, with $(c_i, d_i) = (7075430, 3538809), (26324, 17901)$, we see that equation (3.1) cannot hold in this case.

Thus we assume that $\ell \geqslant 5$. Let H be the centralizer in $\widehat{S}$ of the subspace generated by $\{e_1, f_1\}$. Then $H \cong Sp_{2\ell-2}(3)$. By (3.4), we have $3^{m-1} \leqslant |Aut(PSp_{2\ell}(3)) : PSp_{2\ell-2}(3)| = 2.3^{\ell^2}(3^{2\ell}-1)/3^{(\ell-1)^2} < 3^{4\ell}$. Hence $2m+1 < 8\ell+3$. As $2m+1 = \ell(2\ell+1)$, it follows that $\ell(2\ell+1) < 8\ell+3$. However as $\ell \geqslant 5$, $\ell(2\ell+1) \geqslant 5(2\ell+1) = 10\ell+5 > 8\ell+3$, a contradiction. Next consider case $\lambda = \lambda_{\ell-1}$. As $dimL(\lambda_{\ell-1}) = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell)$ is odd, $\ell = 6k+2, 6k+3$ or $6k+4$.

Let $v = e_1 \wedge e_2 + \xi f_1 \wedge f_2 + \langle w \rangle \cap w^{\perp}$, where $\xi = \pm 1$. Then v is non-singular in V. Let $N_1 = \langle e_1, e_2, f_1, f_2 \rangle$ be a subspace of N. Since N_1 is non-degenerate, $N = N_1 \perp N_1^{\perp}$. Let H, K be the centralizers in $\widehat{S}$ of $N_1, N_1^{\perp}$, respectively. Then $H \cong Sp_{2(\ell-2)}(3)$, $K \cong Sp_4(3)$, and H, K commute. In the basis $\beta_1 = \{e_1, e_2, f_1, f_2\}$, let

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\xi & 0 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

As $gBg^t = B, hBh^t = B$, and $det(g) = 1 = det(h)$, $g, h \in Sp(V_1)$. Furthermore, g, h are of order 3 and $gh = hg$, the subgroup generated by g and h are elementary abelian of order 9. Since $vg = v$ and $vh = h$, it follows that $E = \langle g, h \rangle \leqslant K_v$. Thus $E \times H \leqslant \widehat{S}_v$, hence $1 + c + d \leqslant |Aut(S) : (E \times H)| = |Aut(PSp_{2\ell}(3)) : (E \times PSp_{2(\ell-2)}(3))| < 3^{8\ell-7}$. Hence $2m+1 < 16\ell - 11$. Since $2m+1 = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell) \geqslant 2\ell^2 - \ell - 2$, we have $2\ell^2 - \ell - 2 < 16\ell - 11$, or equivalent $2\ell^2 - 17\ell + 9 < 0$. As $\ell = 6k+2, 6k+3, 6k+4$, if $\ell > 4$ then $k \geqslant 1$, and so $\ell \geqslant 8$. Then $2\ell^2 - 17\ell + 9 \geqslant 2\ell^2 - 16\ell - (\ell - 8) + 1 = (2\ell - 1)(\ell - 8) + 1 > 0$, a contradiction. Thus $\ell \leqslant 4$.

If $\ell = 4$, then $dimL(\lambda_{\ell-1}) = 27$, by using GAP, equation (3.1) cannot hold. If $\ell = 3$, then $dimL(\lambda_{\ell-1}) = 13$. Using GAP, there is only one orbit of minus points and two orbits of plus points. Hence equation (3.1) holds for both types of points by Corollary 3.7. We are left with case $\ell = 4$, $\lambda = \lambda_1, dimL(\lambda_1) = 41$.

Case $f \geqslant 2$. It follows from case $f = 1$ that if λ is a 3-restricted dominant weight such that $L(\lambda)$ is self-dual and is of odd degree then $dimL(\lambda) \geqslant 2\ell^2 - \ell - 1 - \varepsilon_p(\ell) \geqslant 2\ell^2 - \ell - 2$. As V is an absolutely irreducible $k\widehat{S}$-module, by Propositions 2.41 and 2.39, $2m + 1 = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ of odd degree. Thus $2m + 1 = (dim\Psi)^f \geqslant (2\ell^2 - \ell - 2)^f$. By (3.4), $3^{m-1} \leqslant |Aut(PSp_{2\ell}(3))| \leqslant 3^{f(2\ell^2+\ell+1)}$, so that $2m + 1 < 2f(2\ell^2 + \ell + 1) + 3$. Combining these inequalities, we have $(2\ell^2 - \ell - 2)^f < 2f(2\ell^2 + \ell + 1) + 3$, where $\ell \geqslant 3, f \geqslant 2$. However by induction we see that this inequality cannot happen. The proof is now completed. ■

Let $\widehat{S}$ be a simply connected group of type D_ℓ or ${}^2D_\ell$ over $\mathbf{F}_q$, where $q = p^f$, and $\mathfrak{G}$ be the corresponding simply connected, simple algebraic group over k, such that $S = \mathfrak{G}_\sigma$ for some suitable Frobenius map σ. Let N be the natural module for $\widehat{S}$ with the standard basis β. The representing matrix of the non-degenerate symmetric bilinear form on N has the form $B = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$. From the isomorphisms $\Omega_4^+(q) \cong SL_2(q) \circ SL_2(q), \Omega_4^-(q) \cong L_2(q)$ and $P\Omega_6^\pm(q) \cong L_4^\pm(q)$, we can assume that $\ell \geqslant 4$.

Proposition 3.31 *Assume M is almost simple of type S, where $\widehat{S}$ is simply connected of type D_ℓ or ${}^2D_\ell$ over $\mathbf{F}_{3^f}$. There is an M-orbit on $\mathfrak{C}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{C}_\xi(V)$.

Case $f = 1$. Let $\lambda \in X_3$ be a 3-restricted dominant weight such that $L(\lambda) \cong V$. By inequality (3.4), $3^{m-1} \leqslant |Aut(P\Omega_{2\ell}^\varepsilon(3))| \leqslant 3^{2\ell^2-\ell+2}$. Thus $2m + 1 \leqslant 4\ell^2 - 2\ell + 7$. By Theorem 5.1 and 4.4 in [38], either $\ell \geqslant 4$ and $\lambda = \lambda_{\ell-1}, 2\lambda_\ell$ or $\ell = 4$ and $\lambda = 2\lambda_1, 2\lambda_2$.

If $\lambda = \lambda_{\ell-1}$, then $dimL(\lambda) = 2\ell^2 - \ell$ and $L(\lambda) \cong \wedge^2 N$. Since $dimL(\lambda)$ is odd, ℓ must be odd. Thus $\ell \geqslant 5$. Let $a = e_1 - \xi f_1, b = e_2 + f_2 \in N$ and $z = a \wedge b \in \wedge^2 N$, where $\xi = \pm 1$. Then $\langle z \rangle$ is non-singular in $V = \wedge^2 N$. Also, let N_1 be a subspace of N generated by a and b. Clearly, N_1 is non-degenerate, $dimN_1 = 2$, and $sgn(N_1) = \xi$. (For any $v \in N_1, v = \alpha a + \beta b, Q(v) = \alpha^2 Q(a) + \beta^2 Q(b) = -\xi\alpha^2 + \beta^2 = 0$ has non-zero solutions if and only if $\xi = 1$.) Since $N = N_1 \perp N_1^{\perp}, dimN_1^{\perp} = 2\ell - 2$ and $sgnN = sgnN_1 \cdot sgn(N_1^{\perp})$, it follows from Proposition 2.5.11 [29] that $sgnN_1^{\perp} = \varepsilon\xi$. Since the discriminant $D(Q) \equiv detB = (-1)^{\ell} = -1 \ (\mathrm{mod}(\mathbf{F}^*)^2)$, as ℓ is odd, by Proposition 4.1.6 in [29], $H = \Omega(N_1^{\perp}) \cong \Omega_{2\ell-2}^{\varepsilon\xi}(3) \leqslant \Omega(N)$ centralizes N_1. Hence $H \leqslant M_{\langle z \rangle}$. Therefore, $1 + c + d \leqslant |Aut(P\Omega_{2\ell}^{\varepsilon}(3)) : \Omega_{2\ell-2}^{\varepsilon\xi}(3)| = 43^{2\ell-2}(3^{\ell} - \varepsilon 1)(3^{\ell-1} + \varepsilon\xi 1) \leqslant 3^{4\ell-1}$. Since $\ell \geqslant 5, 3^{m-1} = 3^{(2\ell^2-\ell-3)/2} > 3^{4\ell-1} > 1 + c + d$, a contradiction to (3.4).

Next if $\lambda = 2\lambda_{\ell}$, then $dimL(\lambda) = 2\ell^2 + \ell - 1 - \varepsilon_3(\ell)$ and $V = L(\lambda)$ is the heart of S^2N, that is, $w^{\perp}/(w^{\perp} \cap \langle w \rangle)$, where $w = \sum_{i=1}^{\ell}(e_i \otimes f_i + f_i \otimes e_i) \in S^2N$ if $\widehat{S}$ is of type D_{ℓ} and $w = \sum_{i=1}^{\ell-1}(e_i \otimes f_i + f_i \otimes e_i) + x \otimes x + y \otimes y$ otherwise. Let $z_{\xi} = e_1 \otimes e_1 + \xi f_1 \otimes f_1 + \langle w \rangle \cap w^{\perp}, \xi = \pm 1$, and $N_1 = \langle e_1, f_1 \rangle \leqslant N$. Then z_{ξ} is non-singular in V. Let $N_1 = \langle e_1, f_1 \rangle \leqslant N$ and H be the centralizer of N_1 in $\Omega(N) \cong \Omega_{2\ell}^{\varepsilon}(3)$. Since $sgnN_1^{\perp} = \varepsilon$, $H \cong \Omega_{2\ell-2}^{\varepsilon}(3)$ and H fixes $\langle z_{\xi} \rangle$. Thus $1 + c + d \leqslant |Aut(P\Omega_{2\ell}^{\varepsilon}(3)) : \Omega_{2\ell-2}^{\varepsilon}(3)| \leqslant 12.3^{2\ell-2}(3^{\ell} - \varepsilon 1)(3^{\ell-1} + \varepsilon 1) \leqslant 3^{4\ell}$. If $\ell \geqslant 5$, then $3^{m-1} \geqslant 3^{(2\ell^2+\ell-5)/2} > 3^{4\ell} > 1 + c + d$, contradicts to (3.4). If $\ell = 4$, then $2m + 1 = 35$, hence $m - 1 = 16$. As $3^{m-1} = 3^{16} = 3^{4\ell} > 1 + c + d$, we also get a contradiction to (3.4).

Case $f \geqslant 2$. It follows from case $f = 1$ that if λ is a 3-restricted dominant weight such that $L(\lambda)$ is self-dual and is of odd degree then $dimL(\lambda) \geqslant 2\ell^2 - \ell$. As V is an absolutely irreducible $k\widehat{S}$-module, by Propositions 2.41 and 2.39, $2m + 1 = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ of odd degree. Thus $2m + 1 = (dim\Psi)^f \geqslant (2\ell^2 - \ell)^f$. By (3.4), $3^{m-1} \leqslant |Aut(P\Omega_{2\ell}^{\varepsilon}(3))| \leqslant 3.3^{f(2\ell^2-\ell+1)}$, so that $2m + 1 < 2f(2\ell^2 - \ell + 1) + 5$. Combining these two inequalities, we have $(2\ell^2 - \ell)^f < 2f(2\ell^2 - \ell + 1) + 5$, where $\ell \geqslant 4, f \geqslant 2$. By induction, this cannot happen. ■

Assume that $\widehat{S}$ is simply connected of exceptional type which defines over a field of characteristic 3. Then $\widehat{S}$ is one of the following types: $G_2, F_4, E_6, E_7, E_8, {}^2E_6, {}^3D_4, {}^2G_2$.

Proposition 3.32 *Assume M is almost simple of type S, where $\widehat{S}$ is of exceptional type above and define over $\mathbf{F}_{3^e}$ with $e \geqslant 1$. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold unless $(L, S, \lambda) = (\Omega_7(3), G_2(3), \lambda_i), i = 1, 2,$ or $(F_4(3), \Omega_{25}(3), \lambda_4)$, and M has one or at most two orbits on $\mathfrak{E}(V)$, respectively, so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7 and so they are in Table 1.2 or $(L, S, \lambda) = (\Omega_{77}(3), E_6(3), \lambda_2)$, $(\Omega_{133}, E_7(3), \lambda_1)$ and they are in Table 1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$.

(a) **Case** G_2. By (3.4), $3^{m-1} \leqslant |Aut(G_2(3^e))| \leqslant 3^{15e}$. Thus $2m + 1 \leqslant 30e + 3$. Assume first that $e = 1$. Then $2m + 1 \leqslant 33$. Let $\lambda \in X_3$ with $V \cong L(\lambda)$. From Appendix A.49 in [38], λ is one of the following weights $\lambda_1, \lambda_2, 2\lambda_1, 2\lambda_2$. If λ is $\lambda_i, i = 1, 2$, then $dimV = 7$. In these cases, we have $G_2(3) \leqslant \Omega_7(3)$. By [9], these are maximal embedding and $G_2(3)$ has only one orbit in $\mathfrak{E}(V)$. If $\lambda = 2\lambda_1$ or $2\lambda_2$ then $dimV = 27$. We have $G_2(3) \leqslant \Omega_{27}(3)$. But this is not a maximal embedding as $G_2(3) \leqslant \Omega_7(3) \leqslant \Omega_{27}(3)$, where the last embedding arises from the symmetric square of the natural module for $\Omega_7(3)$. (see Proposition 3.29). Assume that $e \geqslant 2$. By Propositions 2.41 and 2.39, $2m + 1 = (dim\Psi)^e$, for some self-dual irreducible $k\widehat{S}$-module Ψ of odd degree. It follows that $2m + 1 \geqslant 7^e$. Combining with $2m + 1 \leqslant 30e + 3$, we have $e = 2$. Then $dimV = 7^2 = 49$. However $G_2(9)$ is not maximal in $\Omega_{49}(3)$ since $G_2(9) \leqslant \Omega_7(9) \leqslant \Omega_{7^2}(3)$ where the first embedding arises as in previous case, while the second is the twisted tensor product embedding.

(b) **Case** F_4. By (3.4), $3^{m-1} \leqslant |Aut(F_4(3^e))| \leqslant 3^{53e}$. So $2m + 1 \leqslant 106e + 3$. Assume that $e = 1$. From Appendix A.50 in [38], $\lambda = \lambda_4$ and $dimL(\lambda) = 25$. In this case, we have an embedding $F_4(3) \leqslant \Omega_{25}(3)$. By [8], $F_4(3)$ has 5 orbits of points in V. But there are two orbits of singular points, and so there are at most two orbits for each types of non-singular points. Assume that $e \geqslant 2$. we have $2m + 1 \geqslant 25^e$, so that $25^e \leqslant 106e + 3$. But this cannot happen for any $e \geqslant 2$.

(c) **Case** ${}^\varepsilon E_6$. By (3.4), $3^{m-1} \leqslant |Aut({}^\varepsilon E_6(3^e))| \leqslant 3^{79e}$. Thus $2m + 1 \leqslant 158e + 3$.

Assume that $e = 1$. From Appendix $A.51$ in [38], $\lambda = \lambda_2$ and $dimL(\lambda) = 77$. (Note that $dimL(\lambda_1) = dimL(\lambda_6) = 27$ but these modules are not self-dual). In this case, we have ${}^2E_6(3) \leqslant E_6(3) \leqslant \Omega_{77}(3)$, and $V = L(E_6)/Z(L(E_6))$, where $L(E_6)$ is the Lie algebra of E_6 over $\mathbf{F}_3$. If $e \geqslant 2$, then $77^e \leqslant 158e + 3$. However this cannot happen for any $e \geqslant 2$..

(d) **Case** E_7. By (3.4), $3^{m-1} \leqslant |Aut(E_7(3^e))| \leqslant 3^{134e}$. Thus $2m + 1 \leqslant 268e + 3$. Assume that $e = 1$. From Appendix $A.52$ in [38], $\lambda = \lambda_1$ and $dimL(\lambda) = 133$. We have $E_7(3) \leqslant \Omega_{133}(3)$, and $V = L(E_7)$, the Lie algebra of E_7 over $\mathbf{F}_3$. Assume that $e \geqslant 2$. We have $133^e \leqslant 268e + 3$. This cannot happen for $e \geqslant 2$.

(e) **Case** E_8. By (3.4), $3^{m-1} \leqslant |Aut(E_8(3^e))| \leqslant 3^{249e}$. Thus $2m+1 \leqslant 498e+3$. Assume that $e = 1$. From Appendix $A.53$ in [38], $dimL(\lambda) \geqslant 3875 > 498.1 + 3 = 501$. Assume that $e \geqslant 2$. Clearly $3875^e > 498e + 3$ for any $e \geqslant 2$.

(f) **Case** ${}^3D_4$. By (3.4), $3^{m-1} \leqslant |Aut({}^3D_4(3^e))| \leqslant 3^{30e}$. Thus $2m+1 \leqslant 60e+3$. Assume that $e = 1$. From Appendix $A.53$ in [38], $\lambda = 2\lambda_1, 2\lambda_2, 2\lambda_4$ and $dimL(\lambda) = 35$. Since the splitting field for ${}^3D_4(3)$ is $\mathbf{F}_{3^3}$, $dimV = 3.35 = 105 > 63$. Assume that $e \geqslant 2$. Clearly $35^e > 60e + 3$ for any $e \geqslant 2$.

(g) **Case** ${}^2G_2$. By (3.4), $3^{m-1} \leqslant |Aut({}^2G_2(3^{2e+1}))| \leqslant 3^{8(2e+1)}$. Thus $2m+1 \leqslant 32e+19$. By Proposition 2.41, $2m + 1 \geqslant 7^{2e+1}$, and so $7^{2e+1} \leqslant 32e + 19$. This cannot happen. ■

Embedding of Sporadic groups

Let $\widehat{S}$ be a sporadic simple group. Define $g_3(S) := 2log_3(|Aut(S)|) + 4$.

Lemma 3.33 *Let S be a simple sporadic group. The minimal degrees of irreducible faithful representations of S and its covering groups over $\mathbf{F}_3$ are given in Table 3.4 together with the value of the function $g_3(S)$.*

Proposition 3.34 *Assume M is almost simple of type S, where S is a simple sporadic group. There is an M-orbit on $\mathfrak{E}_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Table 3.4: Minimal degrees of representations for sporadic groups in characteristic 3.

S	$R_3(S)$	$[g_3(S)]$	S	$R_3(S)$	$[g_3(S)]$
M_{11}	5	20	Suz	64	54
M_{12}	10	26	$2.Suz$	12	54
$2.M_{12}$	6	26	$O'N$	154	54
J_1	56	25	$Co3$	22	53
M_{22}	21	28	$Co2$	23	61
$2.M_{22}$	10	28	Fi_{22}	77	63
J_2	13	29	$2.Fi_{22}$	176	63
$2.J_2$	6	29	HN	133	65
M_{23}	22	33	Ly	651	74
HS	22	37	Th	248	75
$2.HS$	56	37	Fi_{23}	253	82
J_3	18	37	$Co1$	276	82
M_{24}	22	39	$2.Co1$	24	82
McL	21	42	J_4	1333	87
He	51	45	Fi'_{24}	781	106
Ru	378	50	B	4371	144
$2.Ru$	28	50	$2.B$	96256	144
M	196882	229			

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}_\xi(V)$. Then by (3.4), $|Aut(S)| \geqslant 3^{m-1}$. It follows that $2m + 1 \leqslant 2log_3(|Aut(S)|) + 3 = g(S)$. By Lemma 3.33 and [19], we need to consider the following cases: $(S, dimV) = (M_{22}, 21), (McL, 21), (Co_2, 23)$. If $(S, dimV) = (M_{22}, 21)$, then $M_{22} < A_{22} < \Omega^\varepsilon_{21}(3)$, hence $N_{\overline{G}}(\overline{S})$ is not maximal in G. If $(S, dimV) = (McL, 21)$, then S has two orbits with representatives $\langle x\rangle, \langle y\rangle$ and stabilizers $S_{\langle x\rangle} = L_3(4) : 2_2$ and $S_{\langle y\rangle} = M_{11}$ which are in different G-orbits. We also have $c_x = 12194, d_x = 10080$ and $c_y = 72809, d_y = 40590$, and we can check that (3.1) cannot hold in any of these cases. For $S : 2$, we also get the same result. Finally if $(S, dimV) = (Co_2, 23)$, then there exist two non-singular points in different G-orbits with representative $\langle x\rangle, \langle y\rangle$ with $S_{\langle x\rangle} = 2^{10} : M_{22} : 2$ and $S_{\langle y\rangle} = HS : 2$ with sizes $|\langle x\rangle S| = 46575 < 3^{m-1} = 3^{10}, |\langle y\rangle S| = 476928$. The parameters for $\langle y\rangle S$ are $c_y = 296450, d_y = 180477$. We can check that (3.1) cannot hold in this case. ■

3.5 $\Omega^{\pm}_{2m}(3)$

3.5.1 Parameters for $\Omega^{\pm}_{2m}(3)$

Assume the hypothesis and notations in section 3.3 with $L = \Omega^{\varepsilon}_{2m}(3), m \geqslant 2$, and $\varepsilon = \pm$. For $\xi \in \mathbf{F}^* = \{\pm 1\}$, we will identify $+1$ or $+$ with $\square$, and -1 or $-$ with $\boxtimes$. Let $\mathfrak{E}^{\varepsilon}_{\xi}(V)$ be the set of all points in V with norm ξ. Take $x \in \mathfrak{E}^{\varepsilon}_{\xi}(V)$, and define

$$\triangle(x) = \mathfrak{E}^{\varepsilon}_{\xi}(V) \cap x^{\perp}, \mathit{\Gamma}(x) = \mathfrak{E}^{\varepsilon}_{\xi}(V) \cap (V - x^{\perp} - \{x\}).$$

Then P has exactly three orbits $\{\langle x \rangle\}, \Delta(x)$ and $\mathit{\Gamma}(x)$ on $\mathfrak{E}^{\varepsilon}_{\xi}(V)$. Recall that $\Omega \leqslant G \leqslant I$.

Lemma 3.35 *Assume $\xi = \pm$. We have*

(i) $|\mathfrak{E}^{\varepsilon}_{\xi}(V)| = \frac{1}{2}3^{m-1}(3^m - \varepsilon)$

(ii) $k = \frac{1}{2}3^{m-1}(3^{m-1} - \varepsilon)$

(iii) $l = 3^{2(m-1)} - 1$

(iv) $\lambda = \frac{1}{2}3^{m-2}(3^{m-1} + \varepsilon)$

(v) $\mu = \frac{1}{2}3^{m-1}(3^{m-2} - \varepsilon)$

(vi) $\sqrt{D} = 4.3^{m-2}$

(vii) $s = 3^{m-2}(\varepsilon + 2)$

(viii) $t = 3^{m-2}(\varepsilon - 2)$

Proof. For any $\xi \in \mathbf{F}^*$, by Lemma 2.11, $|V_{\xi}| = 3^{m-1}(3^m - \varepsilon)$. Hence $|\mathfrak{E}^{\varepsilon}_{\xi}(V)| = \frac{1}{2}|V_{\xi}|$. This proves (i). It also implies that the parameters for G do not depend on ξ. Let $x_{\varepsilon} = e_1 + \varepsilon f_1$ be a non-singular point in V, where $\{e_1, f_1\}$ is taken from a standard basis β for V. Clearly, $x_{\varepsilon}^{\perp}$ has a basis $\{x_{-\varepsilon}\} \cup (\beta - \{e_1, f_1\})$ and $\langle x_{-\varepsilon} \rangle$ is an ε-type point in $x_{\varepsilon}^{\perp}$. For any $y \in x_{\varepsilon}^{\perp} \cap \mathfrak{E}^{\varepsilon}_{\varepsilon}(V)$, $Q(y) = Q(x) = \varepsilon \neq Q(x_{-\varepsilon})$. Hence $x_{\varepsilon}^{\perp} \cap \mathfrak{E}^{\varepsilon}_{\varepsilon}(V)$ is the set of all $-\varepsilon$-type points in $x_{\varepsilon}^{\perp}$. Thus $k = |\mathfrak{E}^{\circ}_{-\varepsilon}(x_{\varepsilon}^{\perp})| = \frac{1}{2}3^{m-1}(3^{m-1} - \varepsilon)$. By Lemma 3.1, we have $l = |\mathfrak{E}^{\varepsilon}(V)| - 1 - k = 3^{2(m-1)} - 1$. To compute λ, take $y_{\varepsilon} = e_2 + \varepsilon f_2 \in \Delta(x_{\varepsilon}) =$

$x_\varepsilon^\perp \cap \mathfrak{E}_\varepsilon^\varepsilon(V)$. Then $\Delta(x_\varepsilon) \cap \Delta(y_\varepsilon) = \langle x_\varepsilon, y_\varepsilon\rangle^\perp \cap \mathfrak{E}_\varepsilon^\varepsilon(V)$. As $\langle x_\varepsilon, y_\varepsilon\rangle^\perp$ has a basis consisting of two orthogonal subsets $\{x_{-\varepsilon}, y_{-\varepsilon}\}$ and $\beta - \{e_1, e_2, f_1, f_2\}$ with $sgn(\langle x_{-\varepsilon}, y_{-\varepsilon}\rangle) = -$ and $sgn(\langle \beta - \{e_1, e_2, f_1, f_2\}\rangle) = \varepsilon$, it follows that $sgn(\langle x_\varepsilon, y_\varepsilon\rangle^\perp) = -\varepsilon$. Hence $\lambda = |\Delta(x_\varepsilon) \cap \Delta(y_\varepsilon)| = |\mathfrak{E}_\varepsilon^{-\varepsilon}(\langle x_\varepsilon, y_\varepsilon\rangle^\perp)| = \frac{1}{2}3^{m-2}(3^{m-1}+\varepsilon)$ as $dim\langle x_\varepsilon, y_\varepsilon\rangle^\perp = 2(m-1)$. By Lemma 3.1, $\mu l = k(k-1-\lambda)$, hence $\mu = \frac{1}{2}3^{m-1}(3^{m-2}-\varepsilon)$. The remaining parameters follow easily.■

Corollary 3.36 *Let M be a subgroup of G. Suppose that equation (3.1) holds for some $r \in \{s, t\}$ and for any $x \in \mathfrak{E}^\varepsilon(V)$. Then*

(i) if $(\varepsilon, r) = (+, s)$ or $(-, t)$ then equation (3.1) has the form

$$c - 2d = \varepsilon 3^{m-1} - 1. \tag{3.8}$$

(ii) If $(\varepsilon, r) = (+, t)$ or $(-, s)$ then equation (3.1) has the form

$$(\varepsilon 3^{m-1} + 1)(\varepsilon 3^m - 3 + c - 2d) = 4c \tag{3.9}$$

(iii) If $\varepsilon = +$ then

$$1 + c + d \geqslant 3^{m-1} \tag{3.10}$$

(iv) If $\varepsilon = -$ then

$$1 + c + d \geqslant \frac{3^{m-1}+3}{2} \geqslant 3^{m-2}. \tag{3.11}$$

Proof. We can write equation (3.1) in the form

$$l = (r+1)c - d\frac{rl}{k}. \tag{3.12}$$

If ε, r are as in (i) then $r = \varepsilon 3^{m-1}$ and $\frac{rl}{k} = 2(\varepsilon 3^{m-1}+1)$. Substitute all these to equation (3.12), we have $(\varepsilon 3^{m-1}-1)(\varepsilon 3^{m-1}+1) = (\varepsilon 3^{m-1}+1)c - 2d(\varepsilon 3^{m-1}+1) = (\varepsilon 3^{m-1}+1)(c-2d)$. Hence (i) follows. If ε and r are as in (ii), then $r = -\varepsilon 3^{m-2}$ and $\frac{rl}{k} = -2(\varepsilon 3^{m-1}+1)/3$.

Now substitute these parameters into equation (3.12) and multiply both sides by 3, we have $3(\varepsilon 3^{m-1}-1)(\varepsilon 3^{m-1}+1) = 4c - (\varepsilon 3^{m-1}+1)c + 2d(\varepsilon 3^{m-1}+1)$, and (ii) follows. Let $A = 1+c+d$. If $(\varepsilon, r) = (+, s)$ then $c = 2d+3^{m-1}-1$. Then $A = 3^{m-1}+3d \geqslant 3^{m-1}$. Next, if $(\varepsilon, r) = (+, t)$, then $2d(3^{m-1}+1) = (3^{m-1}-3)c+(3^{m-1}+1)(3^m-3) \geqslant (3^{m-1}+1)(3^m-3)$. Thus $d \geqslant \frac{1}{2}(3^m-3)$, and $A = 1+c+d \geqslant 1+\frac{1}{2}(3^m-3) = \frac{1}{2}(3^m-1) \geqslant 3^{m-1}$. This proves (iii). If $(\varepsilon, r) = (-, t)$, then $2d-c = 3^{m-1}+1$. Hence $d = \frac{1}{2}(c+1+3^{m-1}) \geqslant \frac{1}{2}(3^{m-1}+1)$, and $A = 1+c+d \geqslant 1+\frac{1}{2}(3^{m-1}+1) = \frac{1}{2}(3^{m-1}+3) \geqslant 3^{m-2}$. Finally, if $(\varepsilon, r) = (-, s)$, then $(3^{m-1}+3)c = (3^m+3)(3^{m-1}-1)+2d(3^{m-1}-1) \geqslant (3^m+3)(3^{m-1}-1)$. Thus $c \geqslant (3^m+3)(3^{m-1}-1)/(3^{m-1}+3)$, and $A = 1+c+d \geqslant 3^{m-2}(3^m+1)/(3^{m-2}+1)$. To finish the proof, we need to show that $3^{m-2}(3^m+1)/(3^{m-2}+1) \geqslant \frac{1}{2}(3^{m-1}+3)$. Clear fractions, this inequality is equivalent to $2.3^{m-2}(3^m+1) \geqslant (3^{m-2}+1)(3^{m-1}+3)$. However, it is easy to see that $2.3^{m-2} \geqslant 3^{m-2}+1$ and $3^m+1 \geqslant 3^{m-1}+3$. Now the result follows by multiplying these two inequalities sides by sides. ∎

3.5.2 Permutation characters of maximal subgroups in $\mathcal{C}$

The reducible subgroups $\mathcal{C}_1$

Proposition 3.37 *Assume G is nearly simple primitive rank 3 of type $\Omega^{\varepsilon}_{2m}(3)$ and M is of type $O^{\varepsilon_1}_{\alpha}(3) \perp O^{\varepsilon_2}_{2m-\alpha}(3)$. There is an M-orbit on $\mathfrak{E}^{\varepsilon}(V)$ such that equation (3.1) does not hold unless M is of type $O_1(3) \perp O_{2m-1}(3)$. In this case M is in Table 1.1.*

Proof. As M is of type $O^{\varepsilon_1}_{\alpha}(3) \perp O^{\varepsilon_2}_{2m-\alpha}(3)$, there exists a non-degenerate subspace $W \leqslant V$ of dimension α such that $M = N_G(W)$. Put $W_1 = W, W_2 = W^{\perp}, \varepsilon_i = sgnW_i$. Write $X_i = X(W_i)$, where X ranges over the symbols Ω, S and I. By Lemma 4.1.1 in [29], we have $M_I = I_1 \times I_2$ and $\Omega_1 \times \Omega_2 \leqslant M_{\Omega}$.

Suppose first that $dimW = 2b+1$ is odd. If $b = 0$ or $m-1$ then the Proposition holds. So suppose that $m-2 \geqslant b \geqslant 1$. Let x be a non-singular vector in W_1, with $\rho(x) = \xi$. Arguing as in Proposition 3.10, we have $\langle x \rangle M_{\Omega} = \langle x \rangle M_I$. Thus we need to compute the parameters for M_Ω in L. Since $\Omega_1 \leqslant M_\Omega$ acts transitively on $\mathfrak{E}^\circ_\xi(W_1)$, and Ω_2 centralizes $\langle x\rangle$, we have $\langle x\rangle M_\Omega = \langle x\rangle\Omega_1$, hence $d = k(W_1), c = l(W_1)$, parameters for $\mathfrak{E}^\circ_\xi(W_1)$. By Lemma 3.8, $d = \frac{1}{2}3^{b-1}(3^b - \xi), c = (3^b - \xi)(3^{b-1} + \xi)$, and hence $c - 2d = \xi.3^b - 1$. If (3.8) holds then $\xi.3^b - 1 = \varepsilon.3^{m-1} - 1$. Hence $b = m - 1$, a contradiction. Thus (3.9) holds. Therefore $(\varepsilon 3^{m-1} + 1)(\varepsilon 3^m - 3 + \xi.3^b - 1) = 4(3^b - \xi)(3^{b-1} + \xi)$, whence $(3^{m-1} + \varepsilon)(3^m - \varepsilon.4 + \varepsilon.\xi.3^b) = 4(3^b - \xi)(3^{b-1} + \xi)$. We will show that $3^{m-1} + \varepsilon > 2(3^{b-1} + \xi) \geqslant 0$ and $3^m - \varepsilon.4 + \varepsilon.\xi.3^b > 2(3^b - \xi) \geqslant 0$, and by multiplying these two inequalities together, we get a contradiction. As $m - 2 \geqslant b, m - 1 \geqslant (b - 1) + 2$, hence $3^{m-1} + \varepsilon - 2(3^{b-1} + \xi) \geqslant 3^2.3^{b-1} - 2.3^{b-1} + \varepsilon - 2.\xi = 7.3^{b-1} + \varepsilon - 2.\xi$. Since $b \geqslant 1$ and $\varepsilon, \xi \in \{\pm\}$, $7.3^{b-1} + \varepsilon - 2.\xi \geqslant 7 + \varepsilon - 2.\xi \geqslant (1 + \varepsilon) + (2 - 2.\xi) + 4 > 0$. Thus $3^{m-1} + \varepsilon > 2(3^{b-1} + \xi)$. For the other inequality, as $m \geqslant b + 2$, $3^m - 4\varepsilon + \varepsilon\xi 3^b - 2(3^b - \xi) \geqslant 9.3^b + (\varepsilon\xi - 2)3^b + 2\xi - 4\varepsilon = (\varepsilon\xi + 1)3^b + (6 + 2\xi - 4\varepsilon) + 6(3^b - 1)$. As the first two terms of the last expression are non-negative and the last one is positive since $3^b - 1 \geqslant 2 > 0$, $(\varepsilon.\xi + 1).3^b + (6 + 2\xi - 4\varepsilon) + 6(3^b - 1) > 0$, hence $3^m + \varepsilon.\xi.3^b > 2(3^b - 2\xi)$.

Secondly, assume that $dimW = 2b$ is even, where $1 \leqslant b \leqslant m - 1$. Note that if $dimW = 2$, then $sgn(W) = -$, otherwise, M_Ω is contained in a stabilizer of a non-singular point, and so it is not maximal in L. Arguing as above, let x be a non-singular vector in W_1 and let $\xi = sgn(W_1)$. Then $\langle x\rangle M_I = \langle x\rangle M_\Omega = \langle x\rangle\Omega(W_1)$, and so $d = k(W_1), c = l(W_1)$, parameters for $\mathfrak{E}^\xi_{Q(x)}(W_1)$. By Lemma 3.35, $d = \frac{1}{2}3^{b-1}(3^{b-1} - \xi)$, and $c = 3^{2b-2} - 1$. We have $c - 2d = \xi 3^{b-1} - 1$. If equation (3.8) holds then $\xi.3^{b-1} - 1 = \varepsilon.3^{m-1} - 1$. This implies that $b = m$, a contradiction. If equation (3.9) holds then $(3^{m-1} + \varepsilon)(3^m - \varepsilon.4 + \varepsilon.\xi.3^{b-1}) = 4(3^{b-1} - \varepsilon)(3^{b-1} + \varepsilon)$. As $m - 1 \geqslant b \geqslant 1$, $3^{m-1} + \varepsilon - 2(3^{b-1} + \varepsilon) \geqslant 3^b - \varepsilon - 2.3^{b-1} = 3^{b-1} - \varepsilon \geqslant 0$, and $3^m - \varepsilon.4 + \varepsilon.\xi.3^{b-1} - 2(3^{b-1} - \varepsilon) \geqslant (1 + \varepsilon\xi)3^{b-1} + 2(3^b - \varepsilon) > 0$. Thus $3^{m-1} + \varepsilon \geqslant 2(3^{b-1} + \varepsilon)$ and $3^m - \varepsilon.4 + \varepsilon.\xi.3^{b-1} > 2(3^{b-1} - \varepsilon)$. Finally, by multiplying these two inequalities, we get a contradiction. This completes the proof. ■

Retain the notation as in Lemma 3.11 and Lemma 3.12.

Proposition 3.38 *Assume G is nearly simple primitive rank 3 of type $\Omega^{\varepsilon}_{2m}(3)$ and M is of type P_α. Then M has at most two orbits in $\mathfrak{E}(V)$ so that $1_P^G \nleqslant 1_M^G$ by Corollary 3.7 and so M is in Table 1.1.*

Proof. It suffices to show that M_Ω has at most 2 orbits on $\mathfrak{E}^{\varepsilon}(V)$. From definition M stabilizes a totally singular subspace W of dimension α. By Witt's Lemma, we can assume that W has a basis $\{e_1, \cdots, e_\alpha\}$, where $e_i's$ are taken from a standard basis for $\Omega^{\varepsilon}_{2m}(3)$. Let $Y = \langle f_1, \cdots, f_\alpha\rangle$ and $X = (W \bigoplus Y)^{\perp}$. By Proposition 2.6 (iv), X has a basis $\{x_1, \ldots, x_s\}$, with $s = 2(m-\alpha)$, such that the matrix representing of the associated bilinear form of Q restricted to X is either $\mathbf{I}_s$ or $diag(\lambda, 1, \cdots, 1)$, according as $D(X) = \square$ or $D(X) = \boxtimes$. Let $\beta = \{e_1, \ldots, e_\alpha, x_1, \ldots, x_s, f_1, \ldots, f_\alpha\}$. Let $U = C_I(W, W^{\perp}/W, V/W^{\perp}), T_0 = N_I(W, Y)$ and $N = N_I(W, Y, X)$. By Lemma 3.12, $U \leqslant L$, $N = T_0 \times I(X)$ and $M_I = U : N$, where $GL_\alpha(3) \cong T_0 \leqslant I(W \bigoplus Y)$. Let $T = \frac{1}{2}T_0$. As $(\mathbf{F}_3^*)^2 = \{1\}$, by Lemma 3.11, $SL_\alpha(3) \cong T = T_0 \cap \Omega(V)$, hence $U(T \times \Omega(X)) \trianglelefteq K$. Fix $\xi \in \{\pm\}$. As $sgn(W \bigoplus Y) = +$, and $sgnV = sgn(W \bigoplus Y).sgn(X)$, we have $sgn(X) = \varepsilon$. Let x be a non-singular vector in X with $\rho(x) = \xi$, and let $a = \xi e_1 + f_1 \in V$. Using the same argument as in Proposition 3.13 together with Lemma 3.35, we have $|\langle x\rangle M_\Omega| \geqslant \frac{1}{2}|x\Omega(X)U| = |\mathfrak{E}^{\varepsilon}_{\xi}(X)||W| = \frac{1}{2}3^{m-1}(3^{m-\alpha} - \varepsilon)$, and $|\langle a\rangle M_\Omega| \geqslant \frac{1}{2}|aTU| = \frac{1}{2}3^{s+\alpha-1}(3^\alpha - 1)$. Since $|\mathfrak{E}^{\varepsilon}_{\xi}| \geqslant |\langle x\rangle M_\Omega| + |\langle a\rangle M_\Omega| \geqslant \frac{1}{2}3^{m-1}(3^{m-\alpha} - \varepsilon) + \frac{1}{2}3^{s+\alpha-1}(3^\alpha - 1) = \frac{1}{2}3^{m-1}(3^m - \varepsilon) = |\mathfrak{E}^{\varepsilon}_{\xi}(V)|$, as $s = 2m - 2\alpha$. It follows that $|\langle x\rangle M_\Omega| = |\langle x\rangle \Omega(X)U|$, $|\langle a\rangle M_\Omega| = |\langle a\rangle TU|$ and $\mathfrak{E}^{\varepsilon}_{\xi}(V) = \langle x\rangle M_\Omega \cup \langle a\rangle M_\Omega$. Thus M_Ω has at most two orbits on $\mathfrak{E}^{\varepsilon}_{\xi}(V)$. ■

The imprimitive subgroups $\mathcal{C}_2$

Proposition 3.39 *Assume G is nearly simple primitive rank 3 of type $\Omega^{\varepsilon}_{2m}(3)$ and M is of type $O_1(3) \wr S_{2m}$. There is an M-orbit on $\mathfrak{E}^{\varepsilon}(V)$ such that equation (3.1) does not hold unless $(n, \varepsilon, r) = (6, -, t)$ or $(n, \varepsilon, \xi, r) = (10, -, \square, t)$, and so M is in Table 1.1.*

Proof. Let $n = 2m$ and let $\{x_1, \ldots, x_n\}$ be an orthonormal basis for V. Argue as in Proposition 3.17, we have $\langle x\rangle M_I = \langle x\rangle M_\Omega$ for any $x \in \{x_1, x_1 + x_2\}$. Therefore it suffices to compute the parameters for M_Ω in L. We also have $|\langle x_1\rangle M_\Omega| = n, d_1 = n-1, c_1 = 0$, and $|\langle x_1+x_2\rangle M_\Omega| = n(n-1), d_2 = n^2-5n+7, c_2 = 4n-8$ by Lemma 3.16. By Proposition 4.2.15 in [29], $\varepsilon = (-)^m$. We consider the following cases:

(i) Case m even. Then $\varepsilon = +$. If $\xi = \boxtimes$, then we can take $x = x_1$ as $Q(x_1) = -1 = \boxtimes$. Then (3.1) becomes $k = -rd_1$. Since $k = \frac{1}{2}3^{m-1}(3^{m-1}-1) > 0, d_1 = 2m-1 > 0$, r must be negative, hence $r = t = 3^{m-2}(\varepsilon.1 - 2) = -3^{m-2}$. Thus $3^m = 4m+1$. This holds only when $m = 2$. As $m \geqslant 4, 3^m > 4m+1$, and so this case cannot happen. If $\xi = \square$, we choose $x = x_1 + x_2$. If $m \geqslant 6$, then $1 + c_2 + d_2 = n(n-1) < 3^{m-1}$. This violates (3.10) so equation (3.1) cannot hold. Thus $m = 4$ and so $c = 24, d = 31$. We can check that (3.1) cannot hold in this case.

(ii) Case m odd. Then $\varepsilon = -$. If $\xi = \boxtimes$, then we can take $x = x_1$. Then (3.1) becomes $k = -rd_1$. Since $k = \frac{1}{2}3^{m-1}(3^{m-1}+1) > 0, d_1 = 2m-1 > 0$, $r = t = 3^{m-2}(\varepsilon.1-2) = -3^{m-1}$. Thus $3^{m-1} = 4m-3$. This holds only when $m = 3$. By [9], we see that M_Ω has only 2 orbits on $\mathfrak{E}^\varepsilon(V)$ so that (3.1) holds in this case. Finally, if $\xi = \square$, we choose $x = x_1+x_2$. Then $2d_2 - c_2 = 2(n(n-7)+11)$. If $m \geqslant 7$, then $1 + c_2 + d_2 = n(n-1) < 3^{m-2}$. This contradicts to (3.11) and hence equation (3.1) cannot hold. Thus since $7 > m \geqslant 3$ and m is odd, $m = 3, 5$. If $m = 3$, using [9], (3.1) holds. Therefore we assume that $m = 5$. We see that $2d_2 - c_2 = 82 = 3^{5-1}+1$. Thus (3.1) holds with $r = t, \xi = \square$. Let $\langle v\rangle$ and $\langle w\rangle$ be two points in V with generators $\sum_{i=1}^5 x_i$ and $\sum_{i=1}^8 x_i$, respectively. Then $\langle v\rangle, \langle w\rangle \in \mathfrak{E}_\xi^-(V)$, and by applying Lemma 3.15, $|\langle v\rangle M_\Omega| = |\mathcal{D}_{10}^5| = 2^4\binom{10}{5}$ and $|\langle w\rangle M_\Omega| = |\mathcal{D}_{10}^8| = 2^7\binom{10}{8}$. As $|\langle x\rangle M_\Omega| + |\langle v\rangle M_\Omega| + |\langle w\rangle M_\Omega = |\mathfrak{E}_\xi^-(V)|$, we conclude that M_Ω has only three orbits on $\mathfrak{E}_\xi^-(V)$ with representatives $\langle x\rangle, \langle v\rangle$ and $\langle w\rangle$.

Parameters for $\langle v\rangle M_\Omega$. By Lemma 3.15, $\langle v\rangle M_\Omega = \mathcal{D}_{10}^5$, the set of all points of length 5. We first determine $d = |\langle v\rangle M_\Omega \cap v^\perp|$. Let $y = \sum_{i=1}^n \alpha_i x_i \in V$, define $supp(y) = \{x_j | \alpha_j \neq 0\}$.

Let $\langle y\rangle$ be a point in $\langle v\rangle M_\Omega \cap v^\perp$ with $y=\sum_{i=1}^{5}\alpha_{i_j}x_{i_j}$. Then $(v,y)=0$ and $|supp(y)|=5$. We consider the following cases:

(1) $supp(y)\cap supp(v)=\emptyset$. Then $y=\sum_{i=6}^{10}\alpha_i x_i$. There are 2^4 such points in this case.

(2) $|supp(y)\cap supp(v)|=1$. Then $(v,y)=\alpha_j\neq 0$, where α_j is the coefficient of the common vector. There are no such points y.

(3) $|supp(y)\cap supp(v)|=2$. Then $y=\alpha_1x_{i_1}+\alpha_2x_{i_2}+\beta_1y_1+\beta_2y_2+\beta_3y_3$, where all coefficients are non-zero, and $x_{i_1},x_{i_2}\in\{1,2,3,4,5\}, y_i\in\{6,7,8,9,10\}$. We have $(v,y)=\alpha_1+\alpha_2=0$. Hence $\{\alpha_1,\alpha_2\}=\{1,-1\}$. There are $2^3\binom{5}{3}$ choices for $\beta_1y_1+\beta_2y_2+\beta_3y_3$ and $\binom{2}{1}\binom{5}{2}$ choices for $\alpha_1x_{i_1}+\alpha_2x_{i_2}$. Thus there are $\frac{1}{2}\binom{2}{1}\binom{5}{2}2^3\binom{5}{3}=2^5.5^2$ points.

(4) $|supp(y)\cap supp(v)|=3$. Then $y=\sum_{j=1}^{3}\alpha_jx_{i_j}+\sum_{j=1}^{2}\beta_jy_j$, and $(v,y)=\sum_{j=1}^{3}\alpha_j=0$. It follow that $\{\alpha_1,\alpha_2,\alpha_3\}=\pm\{1,1,1\}$. There are $2^2\binom{5}{2}$ choices for $\beta_1y_1+\beta_2y_2$, $2\binom{5}{3}$ choices for $\alpha_1x_{i_1}+\alpha_2x_{i_2}+\alpha_3x_{i_3}$, hence there are $\frac{2}{2}\binom{5}{3}2^2\binom{5}{2}=2^4.5^2$ points.

(5) $|supp(y)\cap supp(v)|=4$. Then $y=\sum_{j=1}^{4}\alpha_jx_{i_j}+\beta_1y_1$. We have $(v,y)=\sum_{j=1}^{4}\alpha_j=0$. It follow that $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}=\{1,1,-1,-1\}$. As there are $2\binom{5}{1}$ choices for β_1y_1, $2\binom{4}{2}\binom{5}{4}$ choices for $\sum_{j=1}^{4}\alpha_jx_{i_j}$, there are $\frac{1}{2}\binom{4}{2}\binom{5}{4}.2\binom{5}{1}=6.5^2$ points.

(6) $supp(v)=supp(y)$. Then $y=\sum_{i=1}^{5}\alpha_ix_i$. Then $(v,y)=\sum_{i=1}^{5}\alpha_i=0$. This happens only when $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}=\pm\{1,1,1,1,-1\}$. Thus there are $\binom{5}{1}=5$ points.

Therefore $d=2^4+2^5.5^2+2^4.5^2+6.5^2+5=1371$, and $c=|\mathcal{D}_{10}^5|-1-d=2^4\binom{10}{5}-1-1371=2660$. Clearly, $2d-c=2.1371-2660=82=3^4+1$. Hence (3.1) holds with $r=t,\xi=\square$.

Parameters for $\langle w\rangle M_\Omega$. For any $\langle y\rangle\in\langle w\rangle K\cap w^\perp$, we have $(w,y)=0$, $|supp(y)|=8$ and $|supp(v)\cap supp(y)|\geqslant 6$.

(1) $|supp(y)\cap supp(v)|=6$. Write $y=\sum_{j=1}^{6}\alpha_jx_{i_j}+\sum_{j=1}^{2}\beta_jy_j$. Then $(w,y)=\sum_{j=1}^{6}\alpha_j=0$. It follows that $\{\alpha_1,\cdots\alpha_6\}=\pm\{1,1,1,1,1,1\}$ or $\{1,1,1,-1,-1,-1\}$. There are $\binom{8}{6}2^2$ points in the first case and $\frac{1}{2}\binom{6}{3}\binom{8}{6}2^2$ points in the last case. Thus, there are $\binom{8}{6}2^2+\frac{1}{2}\binom{6}{3}\binom{8}{6}2^2=1232$ points y.

(2) $|supp(y)\cap supp(v)|=7$. Write $y=\sum_{j=1}^{7}\alpha_jx_{i_j}+\beta_1y_1$. Then $(w,y)=\sum_{j=1}^{7}\alpha_j=0$. It

follows that $\{\alpha_1, \cdots, \alpha_7\} = \pm\{1,1,1,1,1,-1,-1\}$. There are $\binom{7}{2}\binom{8}{7}2\binom{2}{1} = 672$ points.

(3) $supp(w) = supp(y)$. Then $y = \sum_{i=1}^{8} \alpha_i x_i$ and $(w, y) = \sum_{i=1}^{8} \alpha_i = 0$. This happens only when $\{\alpha_1, \alpha_2, \cdots, \alpha_8\} = \pm\{1,1,1,1,1,1,1,-1\}$ or $\{1,1,1,1,-1,-1,-1,-1\}$. Then there are $\binom{8}{1} + \frac{1}{2}\binom{8}{4} = 43$ points.

Thus $d = 1232 + 672 + 43 = 1947$, $c = |\mathcal{D}_{10}^{8}| - 1 - d = 3812$, and $2d - c = 82 = 3^4 + 1$. Hence (3.1) holds for $r = t, \xi = \square$. This finishes the proof. ■

Lemma 3.40 *Let (V, Q) be an orthogonal geometry with $dimV = n$. Fix a standard basis for V as in Definition 2.4. Let $x, e \in V$ be a non-singular vector and singular vector respectively. Then*

$$(i)\ |\{v \in V - \{0\} | Q(v) = 0\}| = \begin{cases} 3^{2m} - 1, & \text{if } n = 2m + 1 \\ 3^{2m-1} + 2\varepsilon 3^{m-1} - 1, & \text{if } n = 2m, sgn(V) = \varepsilon. \end{cases}$$

Assume $n = 2m + 1$ and $\xi = \rho(x) = sgn(x^\perp)$.

$$(ii)\ |\{v \in V - \{0\} | Q(v) = 0, (v, x) = \lambda\}| = \begin{cases} 3^{2m-1} - \xi 3^{m-1}, & \text{if } \lambda \neq 0 \\ 3^{2m-1} + 2\xi 3^{m-1} - 1, & \text{if } \lambda = 0. \end{cases}$$

$$(iii)\ |\{v \in V - \{0\} | Q(v) = \xi, (v, e) = \lambda\}| = \begin{cases} 3^{2m-1}, & \text{if } \lambda \neq 0 \\ 3^m(3^{m-1} + \xi), & \text{if } \lambda = 0. \end{cases}$$

$$(iv)\ |\{v \in V - \{0\} | Q(v) = 0, (v, e) = \lambda\}| = \begin{cases} 3^{2m-1}, & \text{if } \lambda \neq 0 \\ 3^{2m-1} - 1, & \text{if } \lambda = 0. \end{cases}$$

Proof. (i) is a corollary of Lemma 2.11 with $q = 3$. For (ii), when $\lambda = 0$, then the left side is the number of singular vectors in $x^\perp$, as $dim(x^\perp) = 2m$, and $sgn(x^\perp) = \xi$, the result follows from (i). When $\lambda \neq 0$, note that $(v, x) = \lambda$ if and only if $(-v, x) = -\lambda$, and there are only two non-zero values for $\lambda \in \mathbf{F}$, the number in the left side is $\frac{1}{2}[3^{2m} - 1 - (3^{2m-1} + 2\varepsilon 3^{m-1} - 1)] = 3^{2m-1} - \xi 3^{m-1}$, this proves (ii). Let $\beta = \{e_1 \cdots, e_m, f_1, \cdots, f_m, a\}$ be the standard basis for V. By Witt's Lemma, we can assume that $e = e_1$. Then $e^\perp = \langle e \rangle \bigoplus W$, where $W = \langle\{e_2, \cdots, e_m, f_1, \cdots, f_m, a\}\rangle$, and $V = \langle e_1 \rangle \bigoplus W \bigoplus \langle f_1 \rangle$. Now, suppose that

$v = \alpha_1 e_1 + w + \alpha_2 f_1 \in V$, where $w \in W, \alpha_i \in \mathbf{F}$ and that $(v,e) = 0, Q(v) = Q(x)$. Then as $(v,e) = \alpha_2 = 0$, $v = \alpha_1 e + w$ and $Q(v) = Q(w) = Q(x) \neq 0$. By Lemma 3.8, there are $3^{m-1}(3^{m-1}+\xi.1)$ such non-singular vectors $w \in W$, where $\rho(w) = \rho(x) = \xi$. As α_1 can take any values in $\mathbf{F}$, there are $3.3^{m-1}(3^{m-1}+\xi.1) = 3^m(3^{m-1}+\xi.1)$ such vectors v. Using the same argument as in (ii), we see that when $(v,e) = \lambda \neq 0$, then $(v,e) = \lambda$ if and only if $(-v,e) = -\lambda$. By Lemma 3.8 there are $3^m(3^m+\xi.1)$ non-singular vectors u in V satisfying conditions $\rho(u) = \rho(x)$, hence there are $\frac{1}{2}[3^m(3^m+\xi.1) - 3^m(3^{m-1}+\xi.1)] = 3^{2m-1}$ such vectors v. This proves (iii). Finally, if $v = \alpha_1 e + w + \alpha_2 \in V$, where $w \in W, \alpha_i \in \mathbf{F}$ and $(v,e) = 0, Q(v) = 0$. Then $v = \alpha_1 e_1 + w$ and $Q(v) = Q(w) = 0$. By Lemma 2.11, there are 3^{2m-2} such w. As α_1 can be chosen arbitrarily, there are 3 choices for α_1. Hence, after excluding 0, we have $3.3^{2m-2} - 1 = 3^{2m-1} - 1$ such v. Using the same argument as above, there are $\frac{1}{2}[3^{2m} - 1 - (3^{2m-1} - 1)] = 3^{2m-1}$ vectors v such that $(v,e) = \lambda \neq 0, Q(v) = 0$. This finishes the proof. ■

Suppose that $\alpha > 1$ and $\mathcal{D}$ is a non-degenerate α-decomposition. By the Main Theorem in [29], if $n \geqslant 13$, then M is not maximal in G unless $\alpha \geqslant 4$.

Proposition 3.41 *Assume G is nearly simple primitive rank 3 of type $\Omega^{\varepsilon}_{2m}(3)$ and M is of type $O_\alpha(3) \wr S_b$, with $\alpha \geqslant 4$ and $b \geqslant 2$. There is an M-orbit on $\mathfrak{E}^{\varepsilon}(V)$ such that equation (3.1) does not hold unless m is odd and $(\varepsilon, r, \alpha, b) = (-, t, m, 2)$. In this case M is in Table 1.1.*

Proof. Assume that V has an orthonormal basis which is the union of orthonormal bases of all V_i. Let $N = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_b$. As $\alpha = dim V_1 \geqslant 4, V_1$ contains non-singular points of both types. Let $\langle x \rangle \in V_1$ be a non-singular point, and $\xi = sgn(V_1) \in \{\pm, \circ\}$. Argue as in Proposition 3.18, we only need to compute the parameters for M_Ω in L. Since S_b permutes the set $\{V_i\}_{i=1}^{b}$ transitively and for $i > 1, \Omega(V_i)$ centralize V_1, $\langle x \rangle M_\Omega = \langle x \rangle N S_b = \langle x \rangle \Omega(V_1) S_b = \mathfrak{E}^{\xi}_{\rho(x)}(V_1) S_b = \cup_{i=1}^{b} \mathfrak{E}^{\xi}_{\rho(x)}(V_i)$. Thus $A = |\langle x \rangle M_\Omega| = b.|\mathfrak{E}^{\xi}_{\rho(x)}(V_1)|$. Also, for

$i > 1$, V_i are all perpendicular to V_1, hence $d = |x^\perp \cap \langle x\rangle M_\Omega| = k_1 + (b-1)|\mathfrak{E}^{\xi}_{\rho(x)}|, c = |\Gamma(x)| = l_1$, where k_1, l_1 are parameters for V_1. We consider the following cases:

(a) $dim V_1 = 2a+1$ and $b \geqslant 3$. Then as $\alpha \geqslant 4$ and α is odd, $\alpha \geqslant 5$, hence $a \geqslant 2$. Also since $n = 2m = (2a+1)b$, it follows that $b = 2b_1$ is even, $b_1 \geqslant 2$, and $m = \alpha b_1$. We have $A = b.|\mathfrak{E}_{\rho(x)}(V_1)| = b.\frac{1}{2}3^a(3^a + \rho(x)) \leqslant b.\frac{1}{2}3^a(3^a+\rho(x)) < b.3^{2a} = b.3^{\alpha-1}$. By Corollary 3.36, it suffices to show that $3^{m-2} > b.3^{\alpha-1}$, or equivalently, $3^{m-2-\alpha+1} = 3^{\alpha b_1 - \alpha - 1} > b$. Since $\alpha > 4, b_1 \geqslant 2$, we have $(\alpha-4)(b_1-2) = \alpha b_1 - \alpha - 1 - (\alpha + 4b_1 - 9) \geqslant 0$, and so $\alpha b_1 - \alpha - 1 \geqslant \alpha + 4b_1 - 9$. As $b = 2b_1 \geqslant 4, \alpha \geqslant 5$, $\alpha + 4b_1 - 9 \geqslant 5 + b + b - 9 \geqslant b$. Hence $3^{\alpha b_1 - \alpha - 1} \geqslant 3^b$. The results follows as $3^b > b$ for all $b \geqslant 4$.

(b) $dim V_1 = 2a$ and $b \geqslant 3$. Then $a \geqslant 2$, and $m = ab$. We have $A = b|\mathfrak{E}^{\xi}_{Q(x)}(V_1)| = b.\frac{1}{2}3^{a-1}(3^a - \xi) \leqslant b.\frac{1}{2}3^{a-1}(3^a+1) < b.3^{2a-1}$. We will show that $3^{m-2} \geqslant b.3^{2a-1}$ and hence equation (3.1) cannot hold by Corollary 3.36. This inequality is equivalent to $3^{ab-2a-1} \geqslant b$. As $b \geqslant 3, a \geqslant 2$, we have $(a-2)(b-3) = ab - 2a - 1 - (a + 2b - 7) \geqslant 0$, hence $ab - 2a - 1 \geqslant b + (a+b-7) \geqslant b-2$. Clearly, $3^{b-2} \geqslant b$ for any $b \geqslant 3$. Thus $3^{ab-2a-1} \geqslant b$.

(c) $dim V_1 = 2a$ and $b = 2$. Then $a \geqslant 2, \varepsilon = \xi^2 = +$, and $m = \alpha = 2a$. By Lemma 3.35, $d = k_1 + |\mathfrak{E}^{\xi}_{Q(x)}(V_2)| = \frac{1}{2}3^{a-1}(3^{a-1} - \xi) + \frac{1}{2}3^{a-1}(3^a - \xi), c = l_1 = 3^{2a-2} - 1$, and $c - 2d = -3^{m-1} - 1 + 2\xi 3^{a-1}$. We can check that equations (3.8) and (3.9) cannot hold.

(d) $dim V_1 = \alpha = 2a+1$ is odd and $b = 2$. Then $m = 2a+1$. Let η be the type of x in V_1. By Proposition 4.2.14 in [29], we have $\varepsilon = (-)^{(q-1)m/2} = -$. By Lemma 3.8, $|\langle x\rangle M_\Omega| = 3^{2a} + \eta.3^a, d = k_1 + |\mathfrak{E}_\eta(V_1)| = \frac{1}{2}3^{a-1}(3^a - \eta) + \frac{1}{2}3^a(3^a+\eta), c = l_1 = (3^a - \eta)(3^{a-1} + \eta)$, and $c - 2d = -3^{m-1} - 1$. Then $c - 2d = -3^{m-1} - 1 = \varepsilon.3^{m-1} - 1$. It follows that equation (3.1) holds with $r = t$ for the point $\langle x\rangle \in \mathfrak{E}^-_{Q(x)}(V)$. Let $y_i \in V_i, i = 1, 2$, be non-singular vectors with $Q(y_i) = -Q(x)$. As V_1, V_2 are isometric, y_i are of type $-\eta$ for $i = 1, 2$. Put $y = y_1 + y_2 \in V$. Then $\langle y\rangle \in \mathfrak{E}^-_{Q(x)}(V)$ and $\langle y\rangle M_\Omega = \langle y_1\rangle \Omega(V_1) + \langle y_2\rangle\Omega(V_2)$. Hence $|\langle y\rangle M_\Omega| = 2(1 + k_1 + l_1)^2 = \frac{1}{2}(3^{2a} - \eta.3^a)^2$. If $v = v_1 + v_2 \in \langle y\rangle M_\Omega \cap y^\perp$, then $v_i \in y_i\Omega(V_i)$ and $(v_1, y_1) + (v_2, y_2) = 0$. If $(v_1, y_1) = (v_2, y_2) = 0$, then $v_i \in y_i\Omega(V_i) \cap y_i^\perp$. Thus there are $2k_1^2$ such points in this case. If $(v_1, y_1) = -(v_2, y_2) \neq 0$, then there are $(l_1+1)^2$ such points. Hence $d = 2k_1^2 + (l_1+1)^2$, and $c = |\langle y\rangle M_\Omega| - 1 - d = (l_1+1)^2 - 1 + 4k_1(l_1+1)$. Now $c - 2d = -3^{m-1} - 1 = \varepsilon 3^{m-1} - 1$. Let $z = x + e$, where $e \in V_2$ is a singular vector. Then $Q(z) = Q(x)$, and hence $\langle z\rangle \in \mathfrak{E}^-_{Q(x)}(V)$. We have $\langle z\rangle M_\Omega = (\langle x\rangle\Omega(V_1) + \langle e\rangle\Omega(V_2)) \cup (\langle e'\rangle\Omega(V_1) + \langle x'\rangle\Omega(V_2))$, where e' is a singular vector in V_1, and x' is a non-singular vector in V_2 with $Q(x') = Q(x)$. Thus $|\langle z\rangle M_\Omega| = \frac{1}{2}.2.|\{v \in V_1 | Q(v) = Q(x)\}|.|\{v \in V_1 | Q(v) = 0\}|$. Hence, by Lemma 3.8 and 3.40 $|\langle z\rangle M_\Omega| = (3^{2a} + \xi.3^a)(3^{2a} - 1)$. We compute parameters for $\langle z\rangle M_\Omega$. For any $v = v_1 + v_2 \in V_1 \perp v_2$ which satisfies the following conditions $(v, z) = (v_1, x) + (v_2, e) = 0$, and $v \in \langle z\rangle M_\Omega$. Then one of the following holds:

(i). $(v_1, x) = 0 = (v_2, e)$, $Q(v_1) = Q(x), Q(v_2) = 0$. By Lemma 3.40 and 3.8, there are $\frac{1}{2}|\{v \in x^\perp | Q(v) = Q(x)\}|.|\{v \in e^\perp - \{0\} | Q(v) = 0\}| = \frac{1}{2}(3^{2a-1} - \xi 3^{a-1})(3^{2a-1} - 1)$ points.

(ii). $(v_1, x) = 0 = (v_2, e)$, with $Q(v_1) = 0, Q(v_2) = Q(x)$. There are $\frac{1}{2}|\{v \in x^\perp | Q(v) = 0\}|.|\{v \in e^\perp - \{0\} | Q(v) = Q(x)\}| = \frac{1}{2}(3^{2a-1} + 2\eta 3^{a-1} - 1)(3^{2a-1} + \xi 3^a)$ such points.

(iii). $(v_1, x) = -(v_2, e) \neq 0$, with $Q(v_1) = Q(x), Q(v_2) = 0$. There are $\frac{2}{2}|\{v \in V | Q(v) = Q(x), (v, x) = 1\}|.|\{v \in V - \{0\} | Q(v) = 0, (v, e) = -1\}| = (3^{2a-1} + 2\eta 3^{a-1}).3^{2a-1}$ points.

(iv). $(v_1, x) = -(v_2, e) \neq 0$ with $Q(v_1) = 0, Q(v_2) = Q(x)$. There are $\frac{1}{2}.2.|\{v \in V | Q(v) = 0, (v, x) = 1\}|.|\{v \in V\} | Q(v) = Q(x), (v, e) = -1\}| = (3^{2a-1} - \xi 3^{a-1}).3^{2a-1}$ such points.

Thus $d = \frac{1}{2}(3^{2a-1} - \eta 3^{a-1})(3^{2a-1} - 1) + \frac{1}{2}(3^{2a-1} + 2\eta 3^{a-1} - 1)(3^{2a-1} + \eta 3^a) + (3^{2a-1} + 2\eta 3^{a-1}).3^{2a-1} + (3^{2a-1} - \eta 3^{a-1}).3^{2a-1} = 3^{4a-1} + \eta.3^{3a-1} - \eta.3^{a-1}$, and so $c = |\langle z\rangle M_\Omega| - 1 - d = 2(3^{4a-1} + \eta.3^{3a-1} - \eta.3^{a-1}) - 3^{2a} - 1$. Then $c - 2d = -3^{2a} - 1 = -3^{m-1} - 1 = \varepsilon 3^{m-1} - 1$. We have $|\langle x\rangle M_\Omega| + |\langle y\rangle M_\Omega| + |\langle z\rangle M_\Omega| = (3^{2a} + \eta.3^a) + \frac{1}{2}(3^{2a} - \eta.3^a)^2 + (3^{2a} + \eta.3^a)(3^{2a} - 1) = \frac{1}{2}.3^{2a}(3^{2a} + 1) = \frac{1}{2}.3^{m-1}(3^m + 1) = |\mathfrak{E}^-_{Q(x)}(V)|$. Therefore M_Ω has only three orbits of non-singular points in $\mathfrak{E}^-_{Q(x)}(V)$ with representatives $\langle x\rangle, \langle y\rangle$ and $\langle z\rangle$. Hence (3.1) holds for $r = t$ and for any points in $\mathfrak{E}^-_{Q(x)}(V)$. ■

Proposition 3.42 *Assume G is nearly simple primitive rank 3 of type $\Omega^+_{2m}(3)$ and M is of type $GL_m(3).2$, so that $\mathcal{D}$ is a totally singular m-decomposition. Then M has only one* *orbit of non-singular points on V so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7 and hence M is in Table 1.1.*

Proof. We may choose $e_i \in V_1$ and $f_j \in V_2$ such that $\beta = \{e_1, \cdots, e_m, f_1, \cdots, f_m\}$ is a standard basis for V. By Lemma 4.2.3 in [29], we have $I_{(\mathcal{D})} \cong GL_m(3)$, and hence $M_I = I_{(\mathcal{D})}.S_2 \cong GL_m(3).2$. Also by (4.2.6) and (4.2.7) in [29], we have $M_\Omega = \Omega_{(\mathcal{D})}.2$ when m is even and $M_\Omega = \Omega_{(\mathcal{D})}$ when m is odd, where $\Omega_{(\mathcal{D})} \cong SL_m(3)$. For $\xi = \pm$, let $v_\xi = e_1 + \xi f_1 \in V$. Then v_ξ is non-singular in V. Choose $x \in I_{(\mathcal{D})}$ with $det_{V_1}(x) = -1$, and x fixes v_ξ. Let $y \in M_I$ be the element which interchanges e_i and f_i for all i. Observe that $\langle x, y\rangle$ leaves $\langle v_\xi\rangle$ invariant. Thus $\langle v_\xi\rangle M_I = \langle v_\xi\rangle M_\Omega = \langle v_\xi\rangle \Omega_{(\mathcal{D})}$. Clearly $|\langle v_\xi\rangle \Omega_{(\mathcal{D})}| = \frac{1}{2}3^{m-1}(3^m - 1) = |\mathfrak{E}_\xi^+(V)|$. Therefore M_Ω has only one orbit of non-singular points in $\mathfrak{E}_\xi^+(V)$. Note that when m is odd, M is maximal in G if only if G contains some element that interchanges V_1, V_2 and fixes $\langle v_\xi\rangle$. ■

Proposition 3.43 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^\varepsilon(3)$ and M is of type $O_m(3)^2$, so that $\mathcal{D}$ is non-degenerate and non-isometric, $m \geqslant 5$ is odd. There is an M-orbit on $\mathfrak{E}^\varepsilon(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. By Proposition 4.2.16 in [29], $\varepsilon = (-)^{(3+1)/2} = +$, $M_I = I_1 \times I_2$ and $M_\Omega = S_1 \times S_2$. Let v_ξ be a non-singular vector in V_1 of type ξ. We have $\langle v_\xi\rangle M_I = \langle v_\xi\rangle M_\Omega = \langle v_\xi\rangle \Omega_1 = \mathfrak{E}_\xi(V_1)$. As m is odd, $m = 2a+1$. By Lemma 3.8, we have $d = k(V_1) = \frac{1}{2}3^{a-1}(3^a - \xi)$ and $c = l(V_1) = (3^a - \xi)(3^{a-1} + \xi)$. Then $c - 2d = \xi 3^a - 1$. We see that (3.1) cannot hold ■

The field extension subgroups $\mathcal{C}_3$

Lemma 3.44 *Assume case $\mathbf{O}^\pm$ holds, with $f_\sharp$ a non-degenerate symmetric bilinear form, q odd and $\alpha = 2$. Let $v_\sharp \in V_\sharp$ satisfy $f_\sharp(v_\sharp, v_\sharp) = \lambda \in \mathbf{F}_\sharp^*$, so that $r_{v_\sharp} \in I_\sharp(V_\sharp)$.*

(i) $span_{\mathbf{F}_\sharp}(v_\sharp)$ is a non-degenerate 2-space in V with discriminant $\mu N(\lambda)$, where $\langle\mu\rangle = \mathbf{F}^*$.

(ii)

$$r_{v_\sharp} \in \begin{cases} \Omega & \text{if } \lambda \notin (\mathbf{F}_\sharp^*)^2 \\ S \setminus \Omega & \text{if } \lambda \in (\mathbf{F}_\sharp^*)^2 \end{cases}$$

Proof. This is Lemma 4.3.19 in [29]. ∎

Proposition 3.45 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^+(3)$ and M is of type $O_{2b}^+(3^\alpha)$, where α is a prime divisor of $2m$, and $\frac{2m}{\alpha} = 2b \geqslant 4$. There is an M-orbit on $\mathfrak{E}_\xi^+(V)$ such that equation (3.1) does not hold unless $\alpha = 2$ or $\alpha = 3$. If $\alpha = 2$ then M has only two orbits on $\mathfrak{E}^+(V)$ so that $1_P^G \nleqslant 1_M^G$ by Corollary 3.7; if $\alpha = 3$ then M has three orbits on $\mathfrak{E}_\xi^+(V)$ and equation (3.1) holds for $r = s$ and for all M-orbits on $\mathfrak{E}_\xi^+(V)$, and hence these cases are in Table 1.1.*

Proof. Let $\beta_\sharp = \{e_1, \ldots, e_b, f_1, \ldots, f_b\}$ be a standard basis for $V_\sharp$ over $\mathbf{F}_\sharp$. Define $\phi_\sharp$ to be $\phi_{\beta_\sharp, Q_\sharp}(\nu)$, where $\langle \nu \rangle = \mathrm{Gal}(\mathbf{F}_\sharp/\mathbf{F})$; and $\gamma^\nu = \gamma^3$ for any $\gamma \in \mathbf{F}_\sharp$. Denote by $I, I_\sharp$ the full isometry groups $I(V, Q, F), I(V_\sharp, Q_\sharp, \mathbf{F}_\sharp)$, respectively. Also let $\Omega = \Omega(V, \mathbf{F}, Q), \Omega_\sharp = \Omega(V_\sharp, Q_\sharp, \mathbf{F}_\sharp)$ and $S = S(V, \mathbf{F}, Q)$. We first prove the following:

(1) $Q_\sharp(w\phi_\sharp) = Q_\sharp(w)^\nu$ for any $w \in V_\sharp$;

(2) $\phi_\sharp$ is of order α. Moreover, if α is an odd prime then $\phi_\sharp \in \Omega(V, Q, \mathbf{F})$. If $\alpha = 2$ then $\phi_\sharp \in \Omega,$ if b is even and $\phi_\sharp \in S \setminus \Omega,$ if b is odd;

(3) If $\phi_\sharp \in M_\Omega$, $\langle z \rangle M_\Omega = \{\langle w \rangle \mid Q_\sharp(w) \in \{\gamma, \gamma^\nu, \cdots, \gamma^{\nu^{\alpha-1}}\}\}$, where $z \in V_\sharp, \gamma = Q_\sharp(z)$.

For (1), write $w = \sum_{i=1}^b \lambda_i e_i + \mu_i f_i$, where $\lambda_i, \mu_i \in \mathbf{F}_\sharp$. Then $w\phi_\sharp = \sum_{i=1}^b (\lambda_i e_i + \mu_i f_i)\phi_\sharp = \sum_{i=1}^b (\lambda_i^\nu e_i + \mu_i^\nu f_i)$, and $Q_\sharp(w\phi_\sharp) = \sum_{i=1}^b (\lambda_i \mu_i)^\nu = (\sum_{i=1}^b \lambda_i \mu_i)^\nu = Q_\sharp(w)^\nu$. For (2), let $\beta_n = \{\zeta, \zeta^3, \ldots, \zeta^{3^{\alpha-1}}\}$ be a normal basis of $\mathbf{F}_\sharp$ over $\mathbf{F}$, $\beta_e = \beta_n \otimes \{e_1, \ldots, e_b\}$, $\beta_f = \beta_n \otimes \{f_1, \ldots, f_b\}$, and $W_e = \langle \beta_e \rangle_{\mathbf{F}}, W_f = \langle \beta_f \rangle_{\mathbf{F}}$. We see that W_e, W_f are two maximal totally singular subspaces in V and $\phi_\sharp$ fixes these two subspaces, so that $\phi_\sharp \in N_{I(V)}(W_e, W_f)$. We now determine whether or not $\phi_\sharp \in \Omega(V)$. Let $\beta_i = \beta_n \otimes e_i = \{\zeta e_i, \ldots, \zeta^{3^{\alpha-1}} e_i\}$. Since $(\zeta^{3^j} e_i)\phi_\sharp = \zeta^{3^{j+1}} e_i$ and $(\zeta^{3^{\alpha-1}} e_i)\phi_\sharp = \zeta^{3^\alpha} e_i = \zeta e_i$,

$$(\phi_\sharp)_{\beta_i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Since $det(\phi_\sharp)_{\beta_i} = (-1)^{\alpha-1}$, $det(\phi_\sharp)_{W_e} = (-1)^{(\alpha-1)b}$. Similarly, $det(\phi_\sharp)_{W_f} = (-1)^{(\alpha-1)b}$. Thus $det(\phi_\sharp)_V = ((-1)^{(\alpha-1)b})^2 = 1$, so that $\phi_\sharp \in S$. By Lemma 3.11, $\phi_\sharp \in \Omega(V)$ if and only if $det(\phi_\sharp)_{W_e} \in (\mathbf{F}^*)^2$. Hence (2) follows easily. We have $M_I = I_\sharp\langle\phi_\sharp\rangle \cong I_\sharp.\mathbb{Z}_\alpha$, and $\Omega_\sharp \leqslant M_\Omega \leqslant I_\sharp\langle\phi_\sharp\rangle$. Assume that $\phi_\sharp \in M_\Omega$. Then $\Omega_\sharp\langle\phi_\sharp\rangle \leqslant M_\Omega \leqslant I_\sharp\langle\phi_\sharp\rangle$. By Lemma 2.3, $\langle z\rangle\Omega_\sharp = \langle z\rangle I_\sharp$. As $\langle z\rangle\Omega_\sharp\langle\phi_\sharp\rangle \subseteq \langle z\rangle M_\Omega \subseteq \langle z\rangle I_\sharp\langle\phi_\sharp\rangle = \langle z\rangle\Omega_\sharp\langle\phi_\sharp\rangle$, the equalities hold through out this chain. Hence $\langle z\rangle\Omega_\sharp\langle\phi_\sharp\rangle = \langle z\rangle M_\Omega = \langle z\rangle M_I$. Finally (3) follows by applying (1).

(a) **Case** $\alpha > 3$. Let $z = e_1 + \xi f_1$, where $\xi = \pm 1$. Then $Q_\sharp(z) = \xi \in \mathbf{F}^*$, $Q(z) = TQ_\sharp(z) = T(\xi) = \pm\alpha \in \mathbf{F}^*$, hence $\langle z\rangle$ is a non-singular point in V, and since ξ is invariant under ν, $\langle z\rangle M_\Omega = \langle z\rangle\Omega_\sharp = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) = \xi\}$. By Lemma 2.11, $|\langle z\rangle M_\Omega| = \frac{1}{2}(q^{2b-1} - q^{b-1})$, where $q = 3^\alpha$. We see that $\langle w\rangle \in \langle z\rangle M_\Omega \cap z_V^\perp$ if and only if $w \in V_\sharp, Q_\sharp(w) = Q_\sharp(z) = \xi, f_\sharp(w,z) = \varphi \in \ker T$, where $T = T_{\mathbf{F}_\sharp/\mathbf{F}}$ is the trace map. Write $w = \varphi f_\sharp(z,z)^{-1}z + z_0$, where $z_0 \in z_{V_\sharp}^\perp$. Then $Q_\sharp(z_0) = \xi^{-1}(\xi^2 - \varphi^2) = \xi(1-\varphi^2)$, as $\xi^2 = 1$. Since $T(\pm 1) \neq 0$, $Q_\sharp(z_0) \neq 0$ for any $\varphi \in \ker T$. Consider the classical orthogonal sub-geometry $(z_{V_\sharp}^\perp, \mathbf{F}_\sharp, (Q_\sharp)_{z_{V_\sharp}^\perp})$ of $(V_\sharp, \mathbf{F}_\sharp, Q_\sharp)$ with $\dim z_{V_\sharp}^\perp = 2(b-1)+1$. Let z_+, z_- be plus and minus type vectors in $(z_{V_\sharp}^\perp, \mathbf{F}_\sharp, (Q_\sharp)_{z_{V_\sharp}^\perp})$, respectively. Denote by n_+, n_- the number of $\varphi \in \ker T$ such that $Q_\sharp(z_0) = Q_\sharp(z_+)$ and $Q_\sharp(z_0) = Q_\sharp(z_-)$, accordingly. As $|\ker\mathrm{T}| = 3^{r-1} = \frac{1}{3}q$, $n_+ + n_- = \frac{1}{3}q$. By Lemma 2.11 again, $d = |\langle z\rangle M_\Omega \cap z_V^\perp| = \frac{1}{2}n_+(q^{2b-2} + q^{b-1}) + \frac{1}{2}n_-(q^{2b-2} - q^{b-1}) = \frac{1}{6}q^{2b-1} + \frac{1}{2}(n_+ + n_-).q^{b-1}$. Thus $2d = \frac{1}{3}q^{2b-1} + (n_+ - n_-)q^{b-1}$. Notice that $n_+ - n_- \in \mathbb{Z}$. Suppose that equation (3.8) holds. Then $c - 2d = 3^{m-1} - 1 = \frac{1}{3}q^b - 1$, so $c = 2d + \frac{1}{3}q^b - 1$. As $1 + c + d = \frac{1}{2}(q^{2b-1} - q^{b-1})$, it follows that $2d = \frac{1}{3}q^{2b-1} - \frac{1}{3}(1+2.3^{\alpha-1})q^{b-1}$. Thus $\frac{1}{3}q^{2b-1} - \frac{1}{3}(1+2.3^{\alpha-1}).q^{b-1} = \frac{1}{3}q^{2b-1} + (n_+ - n_-)q^{b-1}$. This equation yields $n_+ - n_- = -\frac{1}{3}(1+2.3^{\alpha-1})$. However, since $\alpha > 3, 1+2.3^{\alpha-1}$ is not divisible by 3, so the right side of above equation is not an integer while the left side is an integer. This contradiction shows that equation (3.8) cannot hold. Suppose that equation (3.9) holds. Then $(3^{m-1}+1)(3^m - 3 + c - 2d) = 4c$ or equivalently, $3^{2m-1} - 3 + (3^{m-1}-3)c = 2(3^{m-1}+1)d$. Substitute $c = \frac{1}{2}(q^{2b-1} - q^{b-1}) - d - 1$, we have $3^{2m-1} - 3^{m-1} + \frac{1}{2}(3^{m-1}-3).q^{b-1}.(q^b - 1) = (3^m - 1)d$, as $q^b = 3^{\alpha b} = 3^m$, $2.3^{m-1}(3^m-1)+(3^{m-1}-3).q^{b-1}(3^m-1) = (3^m-1).2d$, hence $2d = 2.3^{m-1}+q^{b-1}(3^{m-1}-3) = \frac{1}{3}q^{2b-1} + (2.3^{\alpha-1}-3).q^{\alpha-1}$. Combining this with $2d = \frac{1}{3}q^{2b-1} + (n_+ - n_-)q^{b-1}$, we have $n_+ - n_- = 2.3^{\alpha-1} - 3$ and $n_+ + n_- = 3^{\alpha-1}$. Solving this system of equations, we get $n_- = \frac{1}{2}(3 - 3^{\alpha-1})$. As n_- is a non-negative integer, $3 - 3^{\alpha-1} \geqslant 0$, it follows that $3^{\alpha-1} \leqslant 3$ or $\alpha \leqslant 2$. This contradicts to our assumption that $\alpha > 3$. Thus (3.9) cannot hold.

(b) **Case** $r = 3$. Let ζ be a root of $x^3 - x + 1$ in $\overline{\mathbf{F}}$. Then $\langle\zeta\rangle = \mathbf{F}_\sharp^*$, $T(\zeta^2) = 2$ and $\ker T = \langle 1, \zeta\rangle_{\mathbf{F}}$. Let $\xi = \pm 1$ and $x_1 = e_1 + \xi\zeta^2 f_1, x_2 = e_1 + \xi(\zeta^2+1)f_1, x_3 = e_1 + \xi(\zeta^2+2)f_1$. Then $\gamma_i = Q_\sharp(x_i) = \xi(\zeta^2 + i - 1)$ and $Q(x_i) = TQ_\sharp(x_i) = T(\xi(\zeta^2+i-1)) = 2\xi$. Hence $x_i, 1 \leqslant i \leqslant 3$, are non-singular vectors of V and belong to the same $\Omega-$orbit. Let $Q_i = \{\gamma_i, \gamma_i^\nu, \gamma_i^{\nu^2}\}$. We have $\gamma_i = \xi(\zeta^2+i-1), \gamma_i^\nu = \gamma_i^3 = \xi(\zeta^2+\zeta+i), \gamma_i^{\nu^2} = \gamma_i^9 = \xi(\zeta^2+2\zeta+i)$. By (3), $\langle x_i\rangle M_\Omega = \{\langle w\rangle \mid Q_\sharp(w) \in Q_i\}$, and so, by Lemma 2.11, $1 + c_i + d_i = |\langle x_i\rangle M_\Omega| = \frac{3}{2}(q^{2b-1} - q^{b-1})$. To determine d_i, argue as in case (a), $\langle w\rangle \in \langle x_i\rangle M_\Omega \cap x_i{}_V^\perp$ if and only if $w \in V_\sharp, Q_\sharp(w) \in Q_i$ and $f_\sharp(w, x_i) = \varphi \in \ker T$. Write $w = \varphi f_\sharp(x_i,x_i)^{-1}x_i + w_0$, where $w_0 \in x_i{}_{V_\sharp}^\perp$, hence $Q_\sharp(w_0) = Q_\sharp(w) - \gamma_i^{-1}\varphi^2$. Notice that $\zeta \equiv -1(\mathrm{mod}\ (\mathbf{F}_\sharp^*)^2)$.

(i) First orbit $\langle x_1\rangle M_\Omega$. As $(x_1{}^\perp)_{V_\sharp} = \{e_1 - \xi\zeta^2 f_1, e_2, f_2, \dots\}$, $Q_\sharp(e_1 - \xi\zeta^2 f_1) = -\xi\zeta^2$ and $e_1 - \xi\zeta^2 f_1$ is a plus vector in $(x_1{}^\perp)_{V_\sharp}$, it follows that $w_0 \in (x_1{}^\perp)_{V_\sharp}$ is a plus vector if and only if $Q_\sharp(w_0) \equiv -\xi\zeta^2 \equiv -\xi(\mathrm{mod}\ (\mathbf{F}_\sharp^*)^2)$.

(i_1) If $Q_\sharp(w) = \gamma_1 = \xi\zeta^2$, then $Q_\sharp(w_0) = \gamma_i^{-1}(\gamma_i^2 - \varphi^2) = \zeta^2 - \zeta^{-2}\varphi^2$. We need to determine the number of values of φ in $\ker T$ under which $Q_\sharp(w_0)$ is congruence to ξ or

$-\xi$ modulo (mod $(\mathbf{F}_\sharp^*)^2$). If $\varphi \in \{0, \pm\zeta\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^2, \xi\zeta^{12}\}$. Hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm 1, \pm(\zeta+1), \pm(\zeta-1)\}$. Then $Q_\sharp(w_0) \in \{\xi\zeta^5, \xi\zeta^{11}, \xi\zeta\}$, hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. Thus there are $6(q^{2b-2}+q^{b-1})+3(q^{2b-2}-q^{b-1}) = 9q^{2b-2}+3q^{b-1}$ such vectors w_0.

(i_2) If $Q_\sharp(w) = \gamma_1^3 = \xi(\zeta^2+\zeta+1)$, then $Q_\sharp(w_0) = \gamma_1^{-1}(\gamma_1^4 - \varphi^2) = \zeta^2+\zeta+1-\zeta^{-2}\varphi^2$. If $\varphi \in \{0, \pm\zeta, \pm(\zeta+1), \pm(\zeta-1)\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^6, \xi\zeta^{10}, \xi\zeta^4, \xi\zeta^{16}\}$. Hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm 1\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{23}\}$, hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. Thus there are $2(q^{2b-1}+q^{b-1})+7(q^{2b-2}-q^{b-1}) = 9q^{2b-2}-5q^{b-1}$ such vectors w_0.

(i_3) If $Q_\sharp(w) = \gamma_1^9 = \xi(\zeta^2+2\zeta+1)$, then $Q_\sharp(w_0) = \gamma_1^{-1}(\gamma_1^{10} - \varphi^2) = \zeta^2 - \langle +1-\zeta^{-2}\varphi^2$. If $\varphi \in \{0, \pm\zeta, \pm(\zeta+1), \pm(\zeta-1)\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{18}, \xi\zeta^4, \xi\zeta^2, \xi\}$. Hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm 1\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{15}\}$, hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. Thus, there are $2(q^{2b-2}+q^{b-1})+7(q^{2b-2}-q^{b-1}) = 9q^{2b-2}-5q^{b-1}$ such vectors w_0. Since $q = 3^3$, $2d_1 = 9q^{2b-2}+3q^{b-1}+2.9q^{2b-2}-5q^{b-1} = q^{2b-1}-7q^{b-1}$. Now, as $1+c_1+d_1 = \frac{3}{2}(q^{2b-1}-q^{b-1})$, $c_1 = q^{2b-1}+2q^{b-1}-1$ and $c_1-2d_1 = 9q^{b-1}-1 = 3^{3b-1}-1 = 3^{m-1}-1$ as $m = 3b$. Hence equation (3.8) holds.

(ii) Second orbit $\langle x_2\rangle M_\Omega$. As $({x_2}^\perp)_{V_\sharp} = \{e_1 - \xi(\zeta^2+1)f_1, e_2, f_2, \dots\}$, $Q_\sharp(e_1 - \xi(\zeta^2+1)f_1) = -\xi(\zeta^2+1)$ and $e_1 - \xi\zeta^2 f_1$ is a plus vector in $({x_2}^\perp)_{V_\sharp}$, it follows that $w_0 \in ({x_1}^\perp)_{V_\sharp}$ is a plus vector if and only if $Q_\sharp(w_0) \equiv -\xi(\zeta^2+1) \equiv -\xi\zeta^{21} \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$.

(ii_1) $Q_\sharp(w) = \gamma_2 = \xi(\zeta^2+1)$. Then $Q_\sharp(w_0) = \gamma_2^{-1}(\gamma_2^2 - \varphi^2) = \zeta^5(1-\zeta-\varphi^2)$. If $\varphi \in \{0, \pm 1, \pm\zeta\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{21}, \xi\zeta^{19}, \xi\zeta^3\}$. Hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm(\zeta+1), \pm(\zeta-1)\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{20}, \xi\zeta^{22}\}$, hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. Thus there are $5(q^{2b-2}-q^{b-1})+4(q^{2b-2}-q^{b-1}) = 9q^{2b-2}-q^{b-1}$ such vectors w_0.

(ii_2) If $Q_\sharp(w) = \gamma_2^3 = \xi(\zeta^2+\zeta-1)$, then $Q_\sharp(w_0) = \gamma_1^{-1}(\gamma_2^4 - \varphi^2) = \zeta^5(\zeta^2+\zeta+1-\varphi^2)$. If $\varphi \in \{0, \pm 1, \pm(\zeta+1)\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{11}, \xi\zeta^{15}, \xi\zeta^{19}\}$. Hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm\zeta\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{14}\}$. Hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm(\zeta-1)\}$, then $Q_\sharp(w_0) = 0$. Thus there are $5(q^{2b-2}-q^{b-1})+2(q^{2b-2}+q^{b-1})+2q^{2b-2} = 9q^{2b-2}-3q^{b-1}$

such vectors w_0.

(ii_3) $Q_\sharp(w) = \gamma_1^9 = \xi(\zeta^2 + 2\zeta + 2)$. Then $Q_\sharp(w_0) = \gamma_1^{-1}(\gamma_1^{10} - \varphi^2) = \zeta^5(\zeta^2 - \varphi^2)$. If $\varphi \in \{0, \pm 1, \pm(\zeta - 1)\}$, then $Q_\sharp(w_0) \in= \{\xi\zeta^7, \xi\zeta^{17}, \xi\zeta\}$. Hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm(\zeta+1)\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^8\}$, hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm\zeta\}$, then $Q_\sharp(w_0) = 0$. Thus, there are $5(q^{2b-2} - q^{b-1}) + 2(q^{2b-2} + q^{b-1}) + 2q^{2b-2} = 9q^{2b-2} - 3q^{b-1}$ such vectors w_0. Therefore, $2d_2 = 9q^{2b-2} - q^{b-1} + 2.9q^{2b-2} - 3q^{b-1} = q^{2b-1} - 7q^{b-1}$. As $1 + c_2 + d_2 = \frac{3}{2}(q^{2b-1} - q^{b-1})$, $c_2 = q^{2b-1} + 2q^{b-1} - 1$ and $c_2 - 2d_2 = 9q^{b-1} - 1 = 3^{3b-1} - 1 = 3^{m-1} - 1$. Hence equation (3.8) holds.

(iii) Third orbit $\langle x_3 \rangle M_\Omega$. As $({x_3}^\perp)_{V_\sharp} = \{e_1 - \xi(\zeta^2 - 1)f_1, e_2, f_2, \dots\}$, $Q_\sharp(e_1 - \xi(\zeta^2 - 1)f_1) = \xi\zeta^{25}$ and $e_1 - \xi(\zeta^2 - 1)f_1$ is a plus vector in $({x_3}^\perp)_{V_\sharp}$, it follows that $w_0 \in ({x_3}^\perp)_{V_\sharp}$ is a plus vector if and only if $Q_\sharp(w_0) \equiv \xi\zeta^{25} \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$.

(iii_1) If $Q_\sharp(w) = \gamma_3 = \xi\zeta^2$, then $Q_\sharp(w_0) = \gamma_i^{-1}(\gamma_i^2 - \varphi^2) = \zeta^2 - 1 + \zeta\varphi^2$. If $\varphi \in \{0, \pm\zeta, \pm(\zeta + 1), \pm(\zeta - 1)\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{12}, \xi\zeta^6, \xi\zeta^{16}, \xi\zeta^{24}\}$. Hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm 1\}$, then $Q_\sharp(w_0) = \xi\zeta^{11}$, hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. Thus there are $7(q^{2b-2} - q^{b-1}) + 2(q^{2b-2} + q^{b-1}) = 9q^{2b-2} - 5q^{b-1}$ such vectors w_0.

(iii_2) If $Q_\sharp(w) = \gamma_3^3 = \xi(\zeta^2 + \zeta)$, then $Q_\sharp(w_0) = \gamma_3^{-1}(\gamma_3^4 - \varphi^2) = \zeta^2 + \zeta + \zeta\varphi^2$. If $\varphi \in \{0, \pm 1, \pm(\zeta - 1)\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^{10}, \xi\zeta^4, \xi\zeta^8\}$. Hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm\zeta, \pm(\zeta + 1)\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^7, -\xi\}$, hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. Thus there are $5(q^{2b-1} - q^{b-1}) + 4(q^{2b-2} + q^{b-1}) = 9q^{2b-2} - q^{b-1}$ such vectors w_0.

(iii_3) If $Q_\sharp(w) = \gamma_3^9 = \xi(\zeta^2 + 2\zeta)$, then $Q_\sharp(w_0) = \gamma_1^{-1}(\gamma_1^{10} - \varphi^2) = \zeta^2 + \zeta - \zeta\varphi^2$. If $\varphi \in \{0, \pm 1, \pm\zeta\}$, then $Q_\sharp(w_0) \in \{\xi\zeta^4, \xi\zeta^2, \xi\zeta^{12}, \xi\}$. Hence $Q_\sharp(w_0) \equiv \xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. If $\varphi \in \{\pm(\zeta + 1), \pm(\zeta - 1)\}$. Then $Q_\sharp(w_0) \in \{\xi\zeta^3, \xi\zeta^5\}$, hence $Q_\sharp(w_0) \equiv -\xi(\text{mod } (\mathbf{F}_\sharp^*)^2)$. Thus there are $5(q^{2b-2} - q^{b-1}) + 4(q^{2b-2} + q^{b-1}) = 9q^{2b-2} - q^{b-1}$ such vectors w_0. Hence $2d_3 = 9q^{2b-2} - 5q^{b-1} + 2.9q^{2b-2} - q^{b-1} = q^{2b-1} - 7q^{b-1}$. As $1 + c_3 + d_3 = \frac{3}{2}(q^{2b-1} - q^{b-1})$, we have $c_3 = q^{2b-1} + 2q^{b-1} - 1$ and $c_3 - 2d_3 = 9q^{b-1} - 1 = 3^{3b-1} - 1 = 3^{m-1} - 1$ as $m = 3b$. Therefore equation (3.8) holds.

Finally as $\sum_{i=1}^{3} |\langle x_i\rangle M_\Omega| = \frac{9}{2}(q^{2b-1} - q^{b-1}) = \frac{1}{2}(3^{2m-1} - 3^{m-1}) = |\mathfrak{E}_\xi^+(V)|$, M_Ω has exactly there orbits on $\mathfrak{E}_\xi^+(V)$.

(*c*) **Case** $r = 2$. By part (2), if b is even then $\phi_\sharp \in \Omega(V, Q, \mathbf{F})$, where $m = 2b$. If b is odd then $\phi_\sharp \in S \setminus \Omega$. Suppose that b is odd. As $f_\sharp(e_1 - f_1, e_1 - f_1) = 1 \in (\mathbf{F}_\sharp^*)^2$, by Lemma 3.44, $r_{e_1-f_1} \in S \setminus \Omega$. Let $\psi = r_{e_1-f_1}\phi_\sharp$. Then $\psi \in I_\sharp\langle\phi_\sharp\rangle \cap \Omega$ and hence $\psi \in M_\Omega$. It follows that $\Omega_\sharp\langle\psi\rangle \leqslant M_\Omega \leqslant I_\sharp\langle\phi_\sharp\rangle = I_\sharp\langle\psi\rangle$. Now, $Q_\sharp(w\psi) = Q_\sharp(wr_{e_1-f_1}\phi_\sharp) = Q_\sharp(wr_{e_1-f_1})^\nu = Q_\sharp(w)^\nu$ as $r_{e_1-f_1} \in I_\sharp$. Define $\psi_\sharp$ to be ψ if b is odd and $\phi_\sharp$ if b is even. Then for any $z \in V_\sharp$, since $\Omega_\sharp\langle\psi_\sharp\rangle \leqslant M_\Omega \leqslant I_\sharp\langle\phi_\sharp\rangle = I_\sharp\langle\psi_\sharp\rangle$, and $\langle z\rangle I_\sharp = \langle z\rangle\Omega_\sharp$, it follows that $\langle z\rangle M_\Omega = \langle z\rangle\Omega_\sharp\langle\psi_\sharp\rangle = \{\langle w\rangle \mid Q_\sharp(w) \in \{Q_\sharp(z), Q_\sharp(z)^3\}\}$. Let ζ be a root of $x^2 - x - 1$ in $\overline{\mathbf{F}}$. Then $\langle\zeta\rangle = \mathbf{F}_\sharp^*$, $\zeta^4 = -1, (\zeta+1)^2 = -1$. Let $\eta = \zeta + 1, \xi = \pm 1$. Let $x_1 = e_1 - \xi f_1$, and $x_2 = e_1 + \xi\zeta f_1$. We have $Q_\sharp(x_1) = -\xi \in \mathbf{F}_\sharp^*, Q_\sharp(x_2) = \xi\zeta$, and hence $\langle x_1\rangle M_\Omega = \langle x_1\rangle\Omega_\sharp$ and $\langle x_2\rangle M_\Omega = \langle x_2\rangle\Omega_\sharp\langle\psi_\sharp\rangle = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) \in \{\xi\zeta, (\xi\zeta)^3\}\}$. By Lemma 2.11, $|\langle x_1\rangle M_\Omega| = \frac{1}{2}(q^{2b-1} - q^{b-1})$ and $|\langle x_2\rangle M_\Omega| = 2\frac{1}{2}(q^{2b-1} - q^{b-1})$. Now, as $|\langle x_1\rangle M_\Omega| + |\langle x_2\rangle M_\Omega| = \frac{1}{2}(3q^{2b-1} - 3q^{b-1}) = \frac{1}{2}3^{m-1}(3^m - 1) = |\mathfrak{E}^+(V)|$, M_Ω has only two orbits on $\mathfrak{E}^+(V)$.

Proposition 3.46 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^-(3)$ and M is of type $O_{2b}^+(3^\alpha)$, where α is a prime divisor of $2m$, and $\frac{2m}{\alpha} = 2b \geqslant 4$. There is an M-orbit on $\mathfrak{E}_\xi^-(V)$ such that equation (3.1) does not hold unless $\alpha = 3$ or $\alpha = 2$ and $\overline{G} = \overline{I}$. If $\alpha = 3$ then M has three orbits on $\mathfrak{E}_\xi^-(V)$ and equation (3.1) holds for $r = t$ and for all M-orbits on $\mathfrak{E}_\xi^-(V)$; if $\alpha = 2$ and $\overline{G} = \overline{I}$ then M has two orbits on $\mathfrak{E}^-(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7, and hence these cases are in Table 1.1.*

Proof. Let $\beta_\sharp = \{v_1, \ldots, v_{2b}\}$ be an $\mathbf{F}_\sharp$ basis of $V_\sharp$ such that $f_{\beta_\sharp}$ is either I_{2b} or $\mathrm{diag}(\mu, 1, \ldots, 1)$ according as $D_\sharp = D(V_\sharp)$ is a square or non-square, where μ is a generators $\mathbf{F}_\sharp^*$. Define

$$\phi_\sharp = \begin{cases} \phi_{\beta_\sharp, Q_\sharp}(\nu) & \text{if } D_\sharp = \square \\ \phi_{\beta_\sharp, Q_\sharp}(\nu)\mathrm{diag}(\mu, 1, \ldots, 1) & \text{if } D_\sharp = \boxtimes; \end{cases} \quad \text{and } \psi_\sharp = \begin{cases} \phi_\sharp & \text{if } D_\sharp = \square \\ \phi_\sharp^2 & \text{if } D_\sharp = \boxtimes. \end{cases}$$

We first show the following:

(1) $Q_\sharp(w\phi_\sharp) = Q_\sharp(w)^\nu$ for any $w \in V_\sharp$;

(2) if α is odd then $\psi_\sharp \in \Omega(V, Q, \mathbf{F})$ so that $M_\Omega = \Omega_\sharp\langle\psi_\sharp\rangle$. If $\alpha = 2$ then $M_\Omega \leqslant I_\sharp$;

(3) If α is odd then for any $z \in V_\sharp$, $\langle z\rangle M_\Omega = \{\langle w\rangle \mid Q_\sharp(w) \in \{\gamma, \gamma^3, \ldots, \gamma^{3^{\alpha-1}}\}\}$, where $\gamma = Q_\sharp(z)$.

For any $w \in V_\sharp$, write $w = \sum_{i=1}^{2b} \lambda_i v_i$. We have $w\phi_\sharp = \lambda_1^3\lambda v_1 + \sum_{i=2}^{2b} \lambda_i^3 v_i$, where λ is 1 or μ according as $D_\sharp$ is a square or non-square. Now $Q_\sharp(w\phi_\sharp) = (\lambda_1^3\lambda)^2 Q_\sharp(v_1) + \sum_{i=2}^{2b} \lambda_i^6 Q_\sharp(v_i) = -(\lambda_1^2\lambda)^3 - \sum_{i=2}^{2b} \lambda_i^6 = (-\lambda_1^2\lambda - \sum_{i=2}^{2b} \lambda_i^2)^3 = Q_\sharp(w)^\nu$, which gives (1). For (2), assume first that r is odd. Let $\beta_n = \{\zeta, \zeta^3, \ldots, \zeta^{3^{\alpha-1}}\}$ be a normal basis of $\mathbf{F}_\sharp$ over $\mathbf{F}$. For $2 \leqslant i \leqslant 2b$, denote by β_i the set $\beta_n \otimes v_i$ and $\beta_1 = \beta_n \otimes \eta v_1$, where $\eta = 1$ if $D_\sharp = \square$ and η satisfies $\eta^2\lambda = -1$, when $D_\sharp = \boxtimes$. Observe that $det((\phi_\sharp)_{\beta_i}) = 1$ for any $i \geqslant 2$ and $det((\phi_\sharp)_{\beta_1})$ is 1 if $D_\sharp$ is a square and it is -1 if $D_\sharp$ is a non-square. Thus $det(\phi_\sharp) = 1$ or -1, whenever $D_\sharp$ is a square or non-square, accordingly. Hence if $D_\sharp = \square$ then $\psi_\sharp = \phi_\sharp \in S$, otherwise, as $\psi_\sharp = \phi_\sharp^2$, $det(\psi_\sharp) = det(\phi_\sharp)^2 = 1$, we also have $\psi_\sharp \in S$. Since $o(\psi_\sharp) = \alpha$ and $[S : \Omega] = 2$, $\psi_\sharp^2 \in \Omega$. However, as $\langle\psi_\sharp\rangle = \langle\psi_\sharp^2\rangle \leqslant \Omega$, $\psi_\sharp \in \Omega$. It follows from the proof of Proposition 4.3.16 in [29] that $[M_\Omega : \Omega_\sharp] = \alpha$. Since $\psi_\sharp \notin \Omega_\sharp$ and $\psi_\sharp \in I_\sharp\langle\phi_\sharp\rangle \cap \Omega = M_\Omega$, it follows that $M_\Omega = \Omega_\sharp\langle\psi_\sharp\rangle$. Secondly assume that $\alpha = 2$. As $\mathrm{sgn}(V_\sharp) = -$, and $\frac{1}{2}(3^2-1)b = 4b$ is even, by Proposition 2.6, $D_\sharp = \boxtimes$. Take $\eta \in \mathbf{F}_\sharp$ such that $\eta^2 = -1$. Let $\beta_1 = \{\eta v_1, \eta\mu v_1\}$ and $\beta_i = \{v_i, \eta v_i\}$ for $i \geqslant 2$. Then $(\phi_\sharp)_{\beta_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $(\phi_\sharp)_{\beta_i} = diag(1, -1)$ so that $det(\phi_\sharp) = det((\phi_\sharp)_{\beta_1}) \prod_{i=2}^{2b} det((\phi_\sharp)_{\beta_i}) = 1.(-1)^{2b-1} = -1$. Thus $det(\phi_\sharp) = det(\phi_\sharp^3) = -1$. As $[M_\Omega : \Omega_\sharp] = \alpha = 2$, for any $g \in M_\Omega \backslash \Omega_\sharp$, $g \in I_\sharp\langle\phi_\sharp\rangle$, hence $g = g_\sharp\phi_\sharp^i$, where $g_\sharp \in I_\sharp$ and $1 \leqslant i \leqslant 3$ as $o(\phi_\sharp) = 2\alpha = 4$. By part (ii) of Lemma 3.44, any reflections in $V_\sharp$ are in S, and since $I_\sharp$ is generated by reflections, $I_\sharp \leqslant S$. It follows that $1 = det(g) = det(g_\sharp)det(\phi_\sharp^i) = (-1)^i$, which forces i even. Therefore $g = g_\sharp\psi_\sharp^j$ for some $j = 0, 1$. For any $w \in V_\sharp$, by part (1), $Q_\sharp(w\psi_\sharp) = Q_\sharp(w)^{\nu^2} = Q_\sharp(w)$, since $o(\nu) = \alpha = 2$, which means that $\psi_\sharp \in I_\sharp$. Thus $M_\Omega \leqslant I_\sharp$. This proves (2). For (3), assume that $\alpha \geqslant 3$. By part (2), $M_\Omega = \Omega_\sharp\langle\psi_\sharp\rangle$. Let $z \in V_\sharp$ be any non-zero vector. Then $\langle z\rangle M_\Omega = \langle z\rangle\Omega_\sharp\langle\psi_\sharp\rangle$. Let $g = g_\sharp\psi_\sharp^i \in \Omega_\sharp\langle\psi_\sharp\rangle$. We have $Q_\sharp(wg) = Q_\sharp((zg_\sharp)\psi_\sharp^i) = Q_\sharp(zg_\sharp)^{\nu^2} = Q_\sharp(z)^{\nu^2}$, by part (1) and the fact that $g_\sharp \in I_\sharp$. Now, as $o(\nu) = \alpha$ is odd, ν^2 is also a generator for $\langle\nu\rangle = \mathrm{Gal}(\mathbf{F}_\sharp/\mathbf{F})$. This proves (3). Observe that when r is odd then for any non-singular vector $z \in V_\sharp$, we have $\langle z\rangle\Omega_\sharp\langle\psi_\sharp\rangle = \langle z\rangle M_\Omega = \langle z\rangle M_I$. Thus it suffices to compute the parameters for M_Ω in L. When $\alpha = 2$ then by Proposition 2.6, $D(Q) = \boxtimes$ as $\frac{1}{2}(3-1)m = 2b$ is even. It follows from Proposition 2.8.2 in [29] that $\ddot{I} = \langle\ddot{r}_\square\rangle$, also $\overline{\Omega} = \overline{S}$ and $|\overline{I} : \overline{\Omega}| = 2$. Therefore either $\overline{G} = \overline{\Omega}$ or $\overline{G} = \overline{I}$.

(a) **Case** $\alpha \geqslant 5$. Let $z \in V_\sharp$ be one of the vectors v_2 or $v_2 + v_3$ so that $Q_\sharp(z) = \xi \in \mathbf{F}^*$. As $Q(z) = TQ_\sharp(z) = T(\xi) = \alpha\xi \in \mathbf{F}^*$, z is a non-singular vector in V. Now, since $Q_\sharp(z) = \xi$ is invariant under ν, by part (3) above, $\langle z\rangle M_\Omega = \{\langle w\rangle \mid Q_\sharp(w) = Q_\sharp(z)\}$. By Lemma 2.11, $|\langle z\rangle M_\Omega| = \frac{1}{2}(q^{2b-1} + q^{b-1})$, where $q = 3^\alpha$. As $\langle w\rangle \in \langle z\rangle M_\Omega \cap z_V^\perp$ if and only if $w \in V_\sharp, Q_\sharp(w) = Q_\sharp(z) = \xi, f_\sharp(w,z) = \varphi \in \mathrm{KerT}$, write $w = \varphi f_\sharp(z,z)^{-1}z + w_0$, where $w_0 \in z_{V_\sharp}^\perp$. Then $Q_\sharp(w_0) = \xi(1-\varphi^2)$. Since $T(\pm 1) \neq 0$, $Q_\sharp(w_0) \neq 0$ for any $\varphi \in \ker T$. Consider the classical orthogonal sub-geometry $(z_{V_\sharp}^\perp, \mathbf{F}_\sharp, (Q_\sharp)_{z_{V_\sharp}^\perp})$ of $(V_\sharp, \mathbf{F}_\sharp, Q_\sharp)$ with $\dim z_{V_\sharp}^\perp = 2(b-1)+1$. Let z_+, z_- be plus and minus type vectors in $(z_{V_\sharp}^\perp, (Q_\sharp)_{z_{V_\sharp}^\perp}, \mathbf{F}_\sharp)$, respectively. Denote by n_+, n_- the number of $\varphi \in \ker T$ such that $Q_\sharp(w_0) = Q_\sharp(z_+)$ and $Q_\sharp(z_0) = Q_\sharp(z_-)$, accordingly. As $|\mathrm{kerT}| = 3^{\alpha-1} = \frac{q}{3}$, $n_+ + n_- = \dfrac{q}{3}$. By Lemma 2.11 again, $d = |\langle z\rangle M_\Omega \cap z_{V_\sharp}^\perp| = \frac{1}{2}n_+(q^{2b-2} + q^{b-1}) + \frac{1}{2}n_-(q^{2b-2} - q^{b-1}) = \frac{1}{6}q^{2b-1} + \frac{1}{2}(n_+ - n_-).q^{b-1}$. Thus $2d = \frac{1}{3}q^{2b-1} + (n_+ - n_-)q^{b-1}$.

Now, suppose that equation (3.8) holds. Then $2d - c = 3^{m-1} + 1 = \frac{1}{3}q^b + 1$. As $1 + c + d = \frac{1}{2}(q^{2b-1} + q^{b-1})$, it follows that $2d = \frac{1}{3}q^{2b-1} + \frac{1}{3}(1 + 2.3^{\alpha-1}).q^{b-1}$. Thus $n_+ - n_- = \frac{1}{3}(1 + 2.3^{\alpha-1})$. However since $\alpha > 1, 1 + 2.3^{\alpha-1}$ is not divisible by 3, so that the right side of above equation is not an integer while the left side is an integer. This contradiction shows that equation (3.8) cannot hold.

Next suppose that equation (3.9) holds. Then $(3^{m-1}-1)(3^m+3-c+2d)=4c$ or equivalently, $3^{2m-1}-3+2d(3^{m-1}-1)c=2(3^{m-1}+3)c$. Substitute $c=\frac{1}{2}(q^{2b-1}+q^{b-1})-d-1$, we have $3^{2m-1}+3^{m-1}+(3^m+1)d=\frac{1}{2}(3^{m-1}+3).q^{b-1}.(q^b+1)$, as $q^b=3^{\alpha b}=3^m$, $2.3^{m-1}(3^m+1)+(3^m+1).d=(3^{m-1}+3).q^{b-1}(3^m+1)$, hence $2d=q^{b-1}(3^{m-1}+3)-2.3^{m-1}=\frac{1}{3}q^{2b-1}+(3-2.3^{r-1}).q^{b-1}$. Combining this with $2d=\frac{1}{3}q^{2b-1}+(n_+-n_-)q^{b-1}$, we have $n_+-n_-=3-2.3^{\alpha-1}$ and $n_++n_-=3^{\alpha-1}$. Solving this system of equations, we get $2n_+=3-3^{\alpha-1}$. As n_+ is non-negative, $3-3^{\alpha-1}\geqslant 0$, it follows that $3^{\alpha-1}\leqslant 3$ or $\alpha\leqslant 2$. This contradicts to our assumption that $\alpha\geqslant 5$. Thus (3.9) cannot hold.

(b) **Case** $\alpha=3$. Let ζ be a root of x^3-x+1 in $\overline{\mathbf{F}}$. Then $\langle\zeta\rangle=\mathbf{F}_\sharp^*$, and $\zeta^3=\zeta-1$. Let z be a non-singular vector in $V_\sharp$, and also a non-singular vectors in V. By part (3), $\langle z\rangle M_\Omega=\{\langle w\rangle \mid Q_\sharp(w)\in\{Q_\sharp(z),Q_\sharp(z)^3,Q_\sharp(z)^9\}\}$. By Lemma 2.11, $1+c_z+d_z=|\langle z\rangle M_\Omega|=\frac{3}{2}(q^{2b-1}+q^{b-1})$. To determine d_z, argue as in case (a), $\langle w\rangle\in\langle z\rangle M_\Omega\cap z_V^\perp$ if and only if $w\in V_\sharp, Q_\sharp(w)\in\{Q_\sharp(z),Q_\sharp(z)^3,Q_\sharp(z)^9\}$ and $f_\sharp(w,x_i)=\varphi\in\ker T$. Write $w=\varphi f_\sharp(z,z)^{-1}z+w_0$, where $w_0\in z_{V_\sharp}^\perp$, hence $Q_\sharp(w_0)=Q_\sharp(w)-Q_\sharp(z)^{-1}\varphi^2$. If equation (3.8) holds, then $2d_z-c_z=3^{m-1}+1$, so that $2d_z=q^{2b-1}+7q^{b-1}$. If equation (3.9) holds then $(3^{m-1}-1)(3^m+3-c_z+2d_z)=4c_z$, and hence $2d_z=q^{2b-1}-9q^{b-1}$. Notice that $\zeta\equiv -1(\mathrm{mod}\ (\mathbf{F}_\sharp^*)^2)$. As $\mathrm{sgn}V_\sharp=-$, by Proposition 2.6, $D_\sharp=\Box$ if and only if $\frac{1}{2}(3^3-1)b=13b$ is odd. Therefore $D_\sharp$ is a square if and only if b is odd. We will show that $2d_z=q^{2b-1}+7q^{b-1}$, so that equation (3.8) holds.

Define $z_-=v_2, z_+=v_2+v_3$ and for any $\xi=\pm$,

$$\begin{cases} z_1 &= \zeta z_\xi \\ z_2 &= \zeta^4 z_{-\xi} \\ z_3 &= \zeta^6 z_\xi. \end{cases}$$

As $Q_\sharp(z_\xi)=\xi$, we have $Q_\sharp(z_i)=\xi(\zeta^2+i-1)$ for $i=1,\cdots,3$. Let $\gamma_i=Q_\sharp(z_i)$ and $Q_i=\{\gamma_i,\gamma_i^3=\xi(\zeta^2+\zeta+i),\gamma_i^9=\xi(\zeta^2-\zeta+i)\}$. Observe that -1 and λ, where λ is any generator for $\mathbf{F}_\sharp^*$, are non-square in $\mathbf{F}_\sharp^*$. We can check that $D\langle v_1, z_\xi\rangle^\perp \equiv -\xi 1 \pmod{(\mathbf{F}_\sharp^*)^2}$, $\mathrm{sgn}\langle v_1, z_\xi\rangle^\perp = \xi(-)^b$ and $Q_\sharp(v_1) \equiv (-1)^{b-1} \pmod{(\mathbf{F}_\sharp^*)^2}$. It follows that for $w_0 \in (z_\xi^\perp)_{V_\sharp}$, w_0 is a $\xi(-)^b$ vector if and only if $Q_\sharp(w_0) \equiv (-1)^b \pmod{(\mathbf{F}_\sharp^*)^2}$, or equivalently, w_0 is a $+$ vector if and only if $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$, where $w_0 \in (z_\xi^\perp)_{V_\sharp}$.

(i) First orbit $\langle z_1\rangle M_\Omega$. We have $Q_\sharp(w_0) = \gamma_1^{3^j} - \gamma_1^{-1}\varphi^2$, where $j = 0, 1, 2$, and $\varphi \in \mathrm{ker}T = \{0, \pm 1, \pm\zeta, \pm(\zeta+1), \pm(\zeta-1)\}$. If $j = 0$ then $Q_\sharp(w_0) \equiv \xi(\zeta^4 - \varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$, and there are 6 values of φ in $\mathrm{ker}T$ such that $\zeta^4 - \varphi^2 \equiv 1 \pmod{(\mathbf{F}_\sharp^*)^2}$ and three values of $\varphi \in \mathrm{ker}T$ such that $\zeta^4 - \varphi^2 \equiv -1 \pmod{(\mathbf{F}_\sharp^*)^2}$. Now if $\zeta^4 - \varphi^2 \equiv 1 \pmod{(\mathbf{F}_\sharp^*)^2}$, then $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$, and if $\zeta^4 - \varphi^2 \equiv -1 \pmod{(\mathbf{F}_\sharp^*)^2}$, then $Q_\sharp(w_0) \equiv -\xi \pmod{(\mathbf{F}_\sharp^*)^2}$. Since $z_1^\perp = z_\xi^\perp$, there are $6(q^{2b-2} + q^{b-1}) + 3(q^{2b-2} - q^{b-1}) = 9q^{2b-2} - 3q^{b-1}$ such vectors w_0 in this case by Lemma 2.11 and argument above. If $j = 1$, then $Q_\sharp(w_0) \equiv \xi(\zeta^8 - \varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$. There are 7 values of $\varphi \in \mathrm{ker}T$ for which $Q_\sharp(w_0) \equiv \xi (\mathrm{mod}(\mathbf{F}_\sharp^*)^2)$ and 2 such values of φ with $Q_\sharp(w_0) \equiv -\xi(\mathrm{mod}(\mathbf{F}_\sharp^*)^2)$, hence there are $9q^{2b-1} + 5q^{b-1}$. If $j = 2$, then $Q_\sharp(w_0) \equiv \xi(\zeta^{20} - \varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$. There are 7 values of $\varphi \in \mathrm{ker}T$ for which $Q_\sharp(w_0) \equiv \xi(\mathrm{mod}(\mathbf{F}_\sharp^*)^2)$ and 2 such values of φ with $Q_\sharp(w_0) \equiv -\xi(\mathrm{mod}(\mathbf{F}_\sharp^*)^2)$, hence there are $9q^{2b-1} + 5q^{b-1}$. Therefore, $2d = 27q^{2b-2} + 7q^{b-1} = q^{2b-1} + 7q^{b-1}$.

(ii) Second orbit $\langle z_2\rangle M_\Omega$. We have $Q_\sharp(w_0) = \gamma_2^{3^j} - \gamma_2^{-1}\varphi^2$, and $z_2^\perp = z_{-\xi}^\perp$. It follows that $w_0 \in (z_2^\perp)_{V_\sharp}$ is a plus vector if and only if $Q_\sharp(w_0) \equiv -\xi \pmod{(\mathbf{F}_\sharp^*)^2}$. Firstly, if $j = 0$ then $Q_\sharp(w_0) \equiv -\xi(\zeta^{16} - \varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$. There are 5 values of φ in $\mathrm{ker}T$ such that $Q_\sharp(w_0) \equiv -\xi \pmod{(\mathbf{F}_\sharp^*)^2}$; and 4 values of such φ such that $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$. Thus there are $5(q^{2b-2} + q^{b-1}) + 4(q^{2b-2} - q^{b-1}) = 9q^{2b-2} + q^{b-1}$ such vectors w_0 in this case. Secondly if $j = 1$ then $Q_\sharp(w_0) \equiv -\xi(\zeta^6 - \varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$. There are 5 values of φ in $\mathrm{ker}T$ such that $Q_\sharp(w_0) \equiv -\xi \pmod{(\mathbf{F}_\sharp^*)^2}$; 2 values of such φ such that $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$; and 2 values of $\varphi \in \mathrm{ker}T$ such that $Q_\sharp(w_0) = 0$. Thus there are $5(q^{2b-2} + q^{b-1}) + 2(q^{2b-2} - q^{b-1}) + 2q^{2b-2} = 9q^{2b-2} + 3q^{b-1}$ such vectors w_0 in this case. Finally if $j = 2$, then $Q_\sharp(w_0) \equiv -\xi(\zeta^2 - \varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$. There are 5 values of φ in $\mathrm{ker}T$ such that $Q_\sharp(w_0) \equiv$

$-\xi \pmod{(\mathbf{F}_\sharp^*)^2}$; 2 values of such φ such that $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$; and 2 values of $\varphi \in \ker T$ such that $Q_\sharp(w_0) = 0$. Thus there are $5(q^{2b-2}+q^{b-1})+2(q^{2b-2}-q^{b-1})+2q^{2b-2} = 9q^{2b-2}+3q^{b-1}$ such vectors w_0 in this case. Therefore $2d = 9q^{2b-2}+q^{b-1}+2(9q^{2b-2}+3q^{b-1}) = q^{2b-1}+7q^{b-1}$.

(iii) Third orbit $\langle z_3\rangle M_\Omega$. We have $Q_\sharp(w_0) = \gamma_3^{3^j} - \gamma_3^{-1}\varphi^2$, and $z_3^\perp = z_\xi^\perp$. It follows that $w_0 \in (z_2^\perp)_{V_\sharp}$ is a plus vector if and only if $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$. Firstly, if $j=0$ then $Q_\sharp(w_0) \equiv \xi(\zeta^{24}-\varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$. There are 7 values of φ in $\ker T$ such that $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$; and 2 values of such φ such that $Q_\sharp(w_0) \equiv -\xi \pmod{(\mathbf{F}_\sharp^*)^2}$. Thus there are $7(q^{2a-2}+q^{a-1})+2(q^{2a-2}-q^{a-1}) = 9q^{2a-2}+5q^{a-1}$ such vectors w_0 in this case. Secondly, if $j=1$ then $Q_\sharp(w_0) \equiv \xi(\zeta^{22}-\varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$. There are 5 values of φ in $\ker T$ such that $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$; 4 values of such φ such that $Q_\sharp(w_0) \equiv -\xi \pmod{(\mathbf{F}_\sharp^*)^2}$. Thus there are $5(q^{2b-2}+q^{b-1})+4(q^{2b-2}-q^{b-1}) = 9q^{2b-2}+q^{b-1}$ such vectors w_0 in this case. Finally if $j=2$, then $Q_\sharp(w_0) \equiv -\xi(\zeta^{16}-\varphi^2) \pmod{(\mathbf{F}_\sharp^*)^2}$. There are 5 values of φ in $\ker T$ such that $Q_\sharp(w_0) \equiv \xi \pmod{(\mathbf{F}_\sharp^*)^2}$; 4 values of such φ such that $Q_\sharp(w_0) \equiv -\xi \pmod{(\mathbf{F}_\sharp^*)^2}$. Thus there are $5(q^{2b-2}+q^{b-1})+4(q^{2b-2}-q^{b-1}) = 9q^{2b-2}+q^{b-1}$ such vectors w_0 in this case. Therefore $2d = 9q^{2b-2}+5q^{b-1}+2(9q^{2b-2}+q^{b-1}) = q^{2b-1}+7q^{b-1}$.

(c) **Case** $r=2$ Assume first that $\overline{G} = \overline{\Omega}$. Then $\Omega \leqslant G \leqslant S$ and as in proof of (2), we have $M_\Omega \leqslant M \leqslant I_\sharp \mathbf{F}^*$. Therefore $\langle z\rangle M_\Omega = \langle z\rangle M = \langle z\rangle \Omega_\sharp$, for any non-singular point z in $V_\sharp$. Since $D_\sharp = \boxtimes$, and so $V_\sharp$ has a basis $\beta_\sharp$ with $f_{\beta_\sharp} = \mathrm{diag}(\zeta, 1, \ldots, 1)$, where $\zeta^2 = \zeta+1$. Let $z_1 = v_2 + (\zeta-1)v_3$ and $z_2 = \eta z_1$, where $\eta \in \mathbf{F}_\sharp^*, \eta^2 = -1$. Then $Q_\sharp(z_i) = (-1)^{i-1}\zeta$, and $Q(z_i) = TQ_\sharp(z_i) = T((-1)^{i-1}\zeta) = (-1)^i \neq 0$. Thus z_i are non-singular vectors in both $V_\sharp$ and V. As $\langle v_1, z_i\rangle_{V_\sharp}^\perp = \langle (\zeta-1)v_2 - v_3, v_4, \ldots, v_{2a}\rangle$, $D\langle v_1, z_i\rangle^\perp = -\zeta \equiv \zeta \pmod{(\mathbf{F}_\sharp^*)^2}$, hence $\mathrm{sgn}\langle v_1, z_i\rangle^\perp = (-)^{\frac{1}{2}(9-1)a-1} = (-)^{4a-1} = -$ and $Q_\sharp(v_1) = -\zeta \equiv \zeta \pmod{(\mathbf{F}_\sharp^*)^2}$. It follows that for any $w_0 \in (z_i^\perp)_{V_\sharp}$, w_0 is a $-$ vector if and only if $Q_\sharp(w_0) \equiv \zeta \pmod{(\mathbf{F}_\sharp^*)^2}$, or equivalently, w_0 is a $+$ vector if and only if $Q_\sharp(w_0) \in (\mathbf{F}_\sharp^*)^2$, where $w_0 \in (z_i^\perp)_{V_\sharp}$. By Lemma 2.11, $1 + c_i + d_i = |\langle z_i\rangle M_\Omega| = \frac{1}{2}(q^{2a-1}+q^{a-1})$. For any $\langle w\rangle \in \langle z_i\rangle M_\Omega \cap z_i^\perp$, write $w =$

$\varphi f_\sharp(z_i,z_i)^{-1}z_i + w_0$, where $w_0 \in z_i{}^{\perp}_{V_\sharp}$ and $\varphi \in \ker T$. Then $f_\sharp(w,z_i) = \varphi$ and $Q_\sharp(w_0) = Q_\sharp(w) - Q_\sharp(z_i)^{-1}\varphi^2 = (-1)^{i-1}\zeta^{-1}(\zeta^2 - \varphi^2)$, as $Q_\sharp(w) = Q_\sharp(z_i) = (-1)^{i-1}\zeta$. As $\ker T = \{0, \pm\eta\}$, if $\varphi = 0$ then $Q_\sharp(w_0) = (-1)^{i-1}\zeta \equiv \zeta \pmod{(\mathbf{F}_\sharp^*)^2}$, and if $\varphi = \pm\eta$, then $Q_\sharp(w_0) = (-1)^{i-1}\zeta^{-1}(\zeta^2 - 1) = (-1)^{i-1}\zeta^{-1}\zeta = (-1)^{i-1} \in (\mathbf{F}_\sharp^*)^2$. Therefore $2d_i = 2(q^{2a-2} + q^{a-1}) + q^{2a-2} - q^{a-1} = 3q^{2a-2} + q^{a-1}$, and so $c_i = 3q^{2a-2} - 1 = 3^{2m-1} - 1$, as $m = 2a, q = 3^2$. It follows that $2d_i - c_i = q^{a-1} + 1 = 3^{m-2} + 1$. Clearly (3.8) cannot hold as $3^{m-2}+1 \neq 3^{m-1}+1$. If (3.9) holds then $(3^{m-1}-1)(3^m+3+2d-c) = 4c_i$. However, $(3^{m-1}-1)(3^m+3+2d-c) = (3^{m-1}-1)(3^m+3+3^{m-2}+1) = (3^{m-1}-1)(3^m+3^{m-2}+4) \neq 4(3^{2m-1}-1)$. Thus, (3.1) cannot hold in this case. Finally, assume that $\overline{G} = \overline{I}$. Then $\phi_\sharp \in G$ and hence $\Omega_\sharp\langle\phi_\sharp\rangle \leqslant H_G$. Thus G has only two orbits on $\mathfrak{E}^-(V)$, and so (3.1) holds. ■

Proposition 3.47 *Assume G is nearly simple primitive rank 3 of type $\Omega^\varepsilon_{2m}(3)$ and M is of type $O_{2a+1}(3^2)$, where $m = 2a+1$. There is an M-orbit on $\mathfrak{E}^\varepsilon_\xi(V)$ such that equation (3.1) does not hold unless $D(V) = \square$ so that $\varepsilon = -$, or $D = \boxtimes$ and $\overline{G} = \overline{I}$, so that $\varepsilon = +$. In these cases M has two orbits on $\mathfrak{E}^\varepsilon_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7, and hence M is in Table 1.1.*

Proof. Let $\beta_\sharp = \{v_1, \ldots, v_{2a+1}\}$ be a basis of $V_\sharp$ over $\mathbf{F}_\sharp$ with $f_{\beta_\sharp} = \lambda I_{2a+1}$, where $\lambda = 1$ if $D_\sharp = D(V_\sharp) = \square$ and $\langle\lambda\rangle = \mathbf{F}_\sharp^*$ if $D_\sharp = \boxtimes$. Define $\phi_\sharp$ as in Proposition 3.20. Then $Q_\sharp(w\phi_\sharp) = Q_\sharp(w)^\nu$. Let $W_i = \mathrm{span}_{\mathbf{F}_\sharp} v_i$. Then W_i are non-degenerate 2-spaces in V with $D(W_i) = -N(\lambda)$, by Lemma 3.44(i) . Also $V = W_1 \perp W_2 \perp \cdots \perp W_{2a+1}$ is a non-degenerate 2-space decomposition of V. Let ζ be a root of $x^2 - x - 1$. Then $\zeta^2 = \zeta + 1$.

(a) **Case** $D = \boxtimes$. By Proposition 2.6, we have $\varepsilon = (-)^{2a} = +$. Also by Propositions 2.7.1 and 2.7.3 in [29], $\overline{S} = \overline{\Omega}$ and $\ddot{I} = \ddot{r}_\square$. It follows that either $\overline{G} = \overline{\Omega}$ or $\overline{G} = \overline{I}$. Assume first that $\overline{G} = \overline{S}$, so that $\Omega \leqslant G \leqslant S$. It follows from Proposition 2.7 that $D(W_i) = \boxtimes$ for each i. Thus $-N(\lambda) \notin (\mathbf{F}^*)^2$, and hence $N(\lambda) \in (\mathbf{F}^*)^2$. Therefore, $\lambda^{\frac{3^2-1}{3-1}} = \lambda^4 \in (\mathbf{F}^*)^2$, so that $\lambda = 1$. Recall that $(\xi v_i)\phi_\sharp = \xi^3\lambda v_i$. Let $\eta \in \mathbf{F}_\sharp^*$ be such that $\eta^2 = -1$. Let $\beta_i = \{v_i, \eta v_i\}$. Then $(\phi_\sharp)_{\beta_i} = \mathrm{diag}(1,-1)$, and hence $\det(\phi_\sharp) = -1$. According to Lemma

3.44(ii), $r_{v_\sharp} \in S$ for any $v_\sharp \in V_\sharp$ with $f_\sharp(v_\sharp, v_\sharp) \neq 0$, and hence $I_\sharp \leqslant S$. Consequently, $M_\Omega \leqslant M \leqslant I_\sharp$. Let $z_1 = v_1 + (\zeta - 1)v_2, z_2 = \eta z_1$. Since $Q_\sharp(z_i) = (-1)^{i-1}\zeta$ and $Q(z_i) = TQ_\sharp(z_i) = T((-1)^{i-1}\zeta) = (-1)^{i-1} \neq 0$, z_i are non-singular vectors in V. As $(z_i^\perp)_{V_\sharp} = \langle(\zeta-1)v_1 - v_2, v_3, \ldots, v_{2a+1}\rangle$, $D((z_i^\perp)_{V_\sharp}) = \det(\mathrm{diag}(-\zeta, \underbrace{1, \ldots, 1}_{2a-1})) = -\zeta \equiv \boxtimes(\bmod (\mathbf{F}_\sharp^*)^2)$, and so by Proposition 2.6(iii), $\mathrm{sgn}(z_i^\perp)_{V_\sharp} = (-1)^{a(9-1)/2-1} = (-)^{4a-1} = -$. Now, as $M_\Omega \leqslant I_\sharp$, and $dim_{\mathbf{F}_\sharp} V_\sharp = 2a+1$, $\langle z_i\rangle M_\Omega = \langle z_i\rangle\Omega_\sharp$, and so as $\langle z_i\rangle$ are minus points, by Lemma 2.11, $1 + c + d = |\langle z_i\rangle\Omega_\sharp| = \frac{1}{2}(q^{2a} - q^a)$, where $q = 3^2$. To compute parameter d, for any $\langle w\rangle \in \langle z_i\rangle M_\Omega \cap (z_i^\perp)_V$, write $w = \varphi f_\sharp(z_i, z_i)^{-1} z_i + w_0$, where $w_0 \in (z_i^\perp)_{V_\sharp}$, and $\varphi \in \ker T$, hence $Q_\sharp(w_0) = (-1)^{i-1}\zeta^{-1}(\zeta^2 - \varphi^2)$. As $T(\pm\zeta) \neq 0$, $Q_\sharp(w_0) \neq 0$ for any $\varphi \in \ker T$. Apply Lemma 2.11 together with the fact that $w_0 \in (z_i^\perp)_{V_\sharp}, \mathrm{sgn}(z_i^\perp)_{V_\sharp} = -$, $\dim(z_i^\perp)_{V_\sharp} = 2a$, and $|\ker T| = 3$, we have $2d = 3(q^{2a-1} + q^{a-1})$. Hence $c = 3q^{2a-1} - 6q^{a-1} - 1 = (q^a + 1)(3q^{a-1} - 1)$ and $c - 2d = -q^a - 1 = -3^{m-1} - 1$. As $D = \boxtimes$, according to Proposition 2.6, $\varepsilon = \mathrm{sgn} V = (-)^{m(3-1)/2-1} = (-)^{m-1} = +$. Clearly, equation (3.8) cannot hold as $c - 2d = -3^{m-1} - 1 \neq 3^{m-1} - 1$. Assume that equation (3.9) holds. Then $(3^{m-1} + 1)(3^m - 3 + c - 2d) = 4c$. As $3^{m-1} = 3^{2a} = q^a$ $3^m = 3^{2a+1} = 3q^a$, and $c - 2d = -q^a - 1$, this equation becomes: $2(q^a + 1)(q^a - 2) = 4(q^a + 1)(3q^{a-1} - 1)$. Dividing both sides by $2(q^a + 1)$ yields, $q^a - 2 = 6q^{a-1} - 2$, or equivalently $q = 6$, a contradiction. Therefore, equation (3.1) cannot hold. Assume that $\overline{G} = \overline{I}$. Then $\phi_\sharp \in G$, and hence G has only two orbits on $\mathfrak{E}^+(V)$, so that equation (3.1) holds.

(b) **Case** $D = \square$. Arguing as previous case we have $D(W_i) = \square$ for all i, and so $\langle\lambda\rangle = \mathbf{F}_\sharp^*$. Take $\beta_i = \{v_i, \lambda v_i\}$. Then $(\phi_\sharp)_{\beta_i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and hence $\det(\phi_\sharp) = 1$, so $\phi_\sharp \in S$. Let $v_\sharp \in V_\sharp$ be such that $f_\sharp(v_\sharp, v_\sharp) = 1$. By Lemma 3.44($ii$), $r_{v_\sharp} \in S \setminus \Omega$. Define

$$\psi_\sharp = \begin{cases} \phi_\sharp & \text{if } \phi_\sharp \in \Omega \\ r_{v_\sharp}\phi_\sharp & \text{otherwise.} \end{cases}$$

Then $\psi_\sharp \in \Omega \cap I_\sharp\langle\phi_\sharp\rangle = M_\Omega$, and $Q_\sharp(w\psi) = Q_\sharp(w)^\nu$, so that $M_\Omega = \Omega_\sharp\langle\psi_\sharp\rangle$. For $i = 1, 2$, put $x_i = \eta^i(v_1 + v_2), y_i = \eta^i(v_1 + \lambda v_2)$. Then $Q_\sharp(x_i) = (-1)^i\lambda, Q_\sharp(y_i) = (-1)^{i+1}$ and $Q(x_i) = T((-1)^i\lambda) = (-1)^i, Q(y_i) = T((-1)^{i+1}) = (-1)^i$, as $T(\lambda) = 1, T(1) = -1$, so that for fixed $i = 1, 2$, $\{x_i, y_i\}$ are non-singular vectors in V which belong to the same Ω-orbit. Observe that $(x_i^\perp)_{V_\sharp} = \langle v_1 - v_2, v_3, \ldots, v_{2a+1}\rangle, (y_i^\perp)_{V_\sharp} = \langle \lambda v_1 - v_2, v_3, \ldots, v_{2a+1}\rangle$, $D((x_i^\perp)_{V_\sharp}) = \det(\mathrm{diag}(\lambda, \underbrace{-\lambda, \ldots, -\lambda}_{2a-1})) = -\lambda^{2a} = \square$, $D((y_i^\perp)_{V_\sharp}) = \det(\mathrm{diag}(1, \underbrace{-\lambda, \ldots, -\lambda}_{2a-1})) = -\lambda^{2a-1} = \boxtimes$, and so $\mathrm{sgn}((x_i^\perp)_{V_\sharp}) = (-)^{4a} = +, \mathrm{sgn}((y_i^\perp)_{V_\sharp}) = (-)^{4a-1} = -$. Since $Q_\sharp(y_i) = (-1)^{i+1}$ is invariant under ν, where $\langle\nu\rangle = \mathrm{Gal}(\mathbf{F}_\sharp/\mathbf{F})$, $\langle y_i\rangle M_\Omega = \langle y_i\rangle\Omega_\sharp$, and so by Lemma 2.11, $|\langle y_i\rangle M_\Omega| = \frac{1}{2}(q^{2a} - q^a)$ as $\langle y_i\rangle$ is a minus point in $V_\sharp$. Next, as $\langle x_i\rangle M_\Omega = \langle x_i\rangle\Omega_\sharp\langle\psi_\sharp\rangle = \{\langle w\rangle \in V_\sharp \mid Q_\sharp(w) \in \{Q_\sharp(x_i), Q_\sharp(x_i)^3\}\}$ and $\langle x_i\rangle$ is a plus point in $V_\sharp$, $|\langle x_i\rangle M_\Omega| = 2\frac{1}{2}(q^{2a} + q^a) = q^{2a} + q^a$. As $D = D(V) = \square$, $\mathrm{sgn}V = (-)^m = -$, hence $|\mathfrak{E}^-(V)| = \frac{1}{2}3^{m-1}(3^m + 1)$, by Lemma 3.35. Now, since $|\langle x_i\rangle M_\Omega| + |\langle y_i\rangle M_\Omega| = q^{2a} + q^a + \frac{1}{2}(q^{2a} - q^a) = \frac{1}{2}q^a(3q^a + 1) = \frac{1}{2}3^{m-1}(3^m + 1) = |\mathfrak{E}^-(V)|$, it follows that M_Ω has exactly two orbits on $\mathfrak{E}^-(V)$. ■

Proposition 3.48 *Assume G is nearly simple primitive rank 3 of type $\Omega^\varepsilon_{2m}(3)$ and M is of type $GU_m(3)$. Then M has only one orbit on $\mathfrak{E}^\varepsilon_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7, and hence M is in Table 1.1.*

Proof. M has only one orbit on $\mathfrak{E}^\varepsilon(V)$ by (3.5.2b) and (3.6.1c) in [34]. ■

The tensor product subgroups $\mathcal{C}_4$

Proposition 3.49 *Assume G is nearly simple primitive rank 3 of type $\Omega^\varepsilon_{2m}(3)$ and M is of type $O^{\varepsilon_1}_{n_1}(3) \otimes O^{\varepsilon_2}_{n_2}(3)$, with $(n_1, \varepsilon_1) \neq (n_2\varepsilon_2)$ and $n_i = 2m_i \geqslant 4$, $i = 1, 2$. There is an M-orbit on $\mathfrak{E}^\varepsilon_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. Write X_i for $X(V_i, f_i)$, where X ranges over the symbols I, S and Ω. Let $v_i \in V_i$ be non-singular vectors taken in some orthogonal bases of V_i, and $x = v_1 \otimes v_2$. Then $\langle x\rangle$

is non-singular V. By (4.4.15) and Lemma 4.4.13(iii) in [29], $S_1 \otimes S_2 \leqslant M_\Omega \leqslant M \leqslant M_I = (I_1 \otimes I_2)\langle z \rangle$, where $z^2 \in I_1 \otimes I_2$. Let N_i be the stabilizer in S_i of v_i. Then $N_i \cong SO_{n_i-1}(3)$ and $N_1 \otimes N_2 \leqslant M_{\Omega\langle x \rangle}$. As $|I_i : N_i| = 2.3^{m_i-1}(3^{m_i} - \varepsilon_i)$, $A = |\langle x \rangle M_I| \leqslant [(I_1 \otimes I_2)\langle z \rangle : N_1 \otimes N_2] \leqslant 8.3^{m_1+m_2-2}(3^{m_1} - \varepsilon_1)(3^{m_2} - \varepsilon_2)$. Since $(3^{m_1} - \varepsilon_1)(3^{m_2} - \varepsilon_2) \leqslant (3^{m_1}+1)(3^{m_2}+1) \leqslant 2.3^{m_1+m_2}$, we deduce that $A \leqslant 16.3^{2m_1+2m_2-2} < 3^{2m_1+2m_2+1}$. Now as $dimV = 2m = 2m_1.2m_2$, $m = 2m_1.m_2$. By inequality (3.10), it suffices to show that $3^{m-1} \geqslant 3^{2m_1+2m_2+1}$. This is equivalent to $2m_1m_2 - 1 \geqslant 2m_1 + 2m_2 + 1$, or $(m_1 - 1)(m_2 - 1) \geqslant 2$. As $n_i \geqslant 4$ for all i, we have $m_i \geqslant 2$ for $i = 1, 2$. If $m_i \geqslant 3$ for some i, then clearly this inequality is correct. Thus we only need to consider the case when $m_1 = m_2 = 2$ and $\varepsilon_1 \neq \varepsilon_2$. This implies that $n = 16$. Without loss, assume that $\varepsilon_1 = +$, so that $\varepsilon_2 = -$. As $\frac{1}{2}(3-1)m_2 = 2$ is even, by Proposition 2.6, $D(V_2) = \boxtimes$, and hence by Proposition 6.3.4 in [29], M_Ω is not maximal in L. In fact, $M_\Omega < Sp_2(3) \otimes Sp_8(3)$. ■

Proposition 3.50 *Assume G is nearly simple primitive rank 3 of type $\Omega^{\varepsilon}_{2m}(3)$ and M is of type $O^{\varepsilon}_{n_1}(3) \otimes O_{n_2}(3)$, with $n_2 = 2m_2 + 1$ odd and $n_1 = 2m_1 \geqslant 4$. There is an M-orbit on $\mathfrak{E}^{\varepsilon}_{\xi}(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. If $n_2 = 3$ then M_Ω is not maximal in L by Proposition 6.3.2[29]. Thus we can assume that $n_2 \geqslant 5$ or $m_2 \geqslant 2$. As $n_1 = 2m_2 \geqslant 4$, $m_1 \geqslant 2$. By Proposition 4.4.17[29], $M_\Omega = \Omega_1 \otimes S_2 \leqslant M_I = I_1 \otimes S_2$. Let $v_i \in V_i$ be non-singular vectors and $x = v_1 \otimes v_2$. Then $\langle x \rangle$ is a non-singular point in V and $\langle x \rangle M_I = \langle x \rangle M_\Omega = \langle v_1 \rangle \Omega_1 \otimes \langle v_2 \rangle S_2 = \langle v_1 \rangle \Omega_1 \otimes \langle v_2 \rangle \Omega_2$. By Lemma 2.11, $A = |\langle x \rangle M_I| \leqslant |v_1\Omega_1|.|v_2\Omega_2| = 3^{m_1-1}(3^{m_1} - \varepsilon).3^{m_2}(3^{m_2} + \xi)$, where $\xi = \rho_{V_2}(v_2)$. As $(3^{m_1} - \varepsilon).(3^{m_2} + \xi) \leqslant (3^{m_1} + 1).(3^{m_2} + 1) \leqslant 2.3^{m_1+m_2} < 3^{m_1+m_2+1}$, we have $A < 3^{2m_1+2m_2}$. In view of inequalities (3.10) and (3.11), we need to show that $3^{m-2} \geqslant 3^{2m_1+2m_2-1}$, or equivalently $m-2 \geqslant 2m_1+2m_2$, where $m = 2m_1m_2+m_1 = \frac{1}{2}dimV$. The last inequality is equivalent to $(m_1-1)(2m_2-1) \geqslant 3$. This is true as $m_i \geqslant 2, i = 1, 2$.■

Let $(U, \mathbf{F}_q, \mathbf{f})$ be a symplectic geometry. Then $\mathbf{f}$ is skew-symmetric and bilinear on U. An ordered pair (u_1, u_2) in U is called a *hyperbolic pair* if $\mathbf{f}(u_1, u_2) = 1$. A subspace of U generated by a hyperbolic pair (u_1, u_2) is called a *hyperbolic plane,* with basis $\{u_1, u_2\}$. By Proposition 2.4.1 in [29], U is of even dimension, say 2ℓ, and has a basis $\beta = \{e_1, f_1, \ldots, e_\ell, f_\ell\}$ such that $\mathbf{f}(e_i, f_i) = \delta_{ij}$ and $\mathbf{f}(e_i, e_j) = \mathbf{f}(f_i, f_j) = 0$, such a basis is called a *standard basis* or *symplectic basis* for U. Assume that q is odd, and $\lambda \in \mathbf{F}^*$. Define $\delta_{\mathbf{f},\beta}(\lambda)$ by

$$e_i\delta_{\mathbf{f},\beta} = \lambda e_i \text{ and } f_i\delta_{\mathbf{f},\beta} = f_i. \tag{3.13}$$

Denote by $\mathcal{H}(U)$ the set of all distinct hyperbolic pairs in U, and define $\mathcal{HP}(U)$ to be the set of all distinct hyperbolic planes in U. The following lemma will give us the cardinalities of these sets.

Lemma 3.51 *Let $(U, \mathbf{F}, \mathbf{f})$ be a symplectic geometry and $\beta = \{e_1, f_1, \ldots, e_\ell, f_\ell\}$ be a standard basis for U. Let $\lambda \in \mathbf{F}^*$. Then*

(1) $|\mathcal{H}(U)| = q^{2\ell-1}(q^{2\ell} - 1)$;

(2) $|\mathcal{HP}(U)| = \dfrac{q^{2\ell-2}(q^{2\ell} - 1)}{q^2 - 1}$;

(3) $|\{(u_1, u_2) \in \mathcal{H}(U) \mid \mathbf{f}(u_1, e_1) = \lambda\}| = q^{4\ell-2}$;

(4) $|\{\langle u_1, u_2\rangle \in \mathcal{HP}(U) \mid \mathbf{f}(u_1, e_1) = \lambda\}| = q^{4\ell-4}$;

Proof. As $\dim U = 2\ell$, there are $q^{2\ell} - 1$ non-zero vector u_1 in U. Since $\dim(u_1^\perp) = 2\ell - 1$, there are $q^{2\ell} - q^{2\ell-1}$ vectors u_2 with $\mathbf{f}(u_1, u_2) \neq 0$. As $|\mathbf{F}^*| = q - 1$, there are $\frac{1}{q-1}(q^{2\ell} - 1)(q^{2\ell} - q^{2\ell-1}) = q^{2\ell-1}(q^{2\ell} - 1)$ pairs (u_1, u_2) with $\mathbf{f}(u_1, u_2) = 1$, which gives (1). By applying (1), in each hyperbolic plane $\langle u_1, u_2\rangle$, there are $q(q^2 - 1)$ hyperbolic pairs. This proves (2). For (3), assume that (u_1, u_2) is a hyperbolic pair in U with $\mathbf{f}(u_1, e_1) = \lambda \neq 0$. We can write $u_1 = -\lambda f_1 + u_0$, where $u_0 \in e_1^\perp$. Since $\dim(e_1^\perp) = 2\ell - 1$, there are $q^{2\ell-1}$ choices for u_0, hence there are the same number of choices for u_1. When u_1 is chosen, clearly, there are $\frac{q^{2\ell} - q^{2\ell-1}}{q-1} = q^{2\ell-1}$ possibilities for u_2 such that $\mathbf{f}(u_1, u_2) = 1$. Hence (3) follows by multiplying these two values. To prove (4), it suffices to show that if $W = \langle u_1, u_2 \rangle$ is a hyperbolic plane with $\mathbf{f}(u_1, e_1) = \lambda$, then there are only q^2 hyperbolic pairs (v_1, v_2) in W satisfying $\mathbf{f}(v_1, e_1) = \lambda$. Write $v_1 = \zeta_1 u_1 + \beta_1 u_2$ and $v_2 = \zeta_2 u_1 + \beta_2 u_2$. It follows from $\mathbf{f}(v_1, v_2) = 1$ that

$$\zeta_1 \beta_2 - \zeta_2 \beta_1 = 1. \tag{3.14}$$

Also, as $\mathbf{f}(u_1, e_1) = \lambda$, and $v_1 = \zeta_1 u_1 + \beta_1 u_2$, we have $\mathbf{f}(v_1, e_1) = \zeta_1 \mathbf{f}(u_1, e_1) + \beta_1 \mathbf{f}(u_2, e_1) = \lambda\zeta_1 + \beta_1\mu$, where $\mu = \mathbf{f}(u_2, e_1)$. Now, since $\mathbf{f}(v_1, e_1) = \lambda$,

$$\lambda\zeta_1 + \beta_1\mu = \lambda. \tag{3.15}$$

If $\mu = 0$, then $\zeta_1 = 1$ and $\beta_2 = 1 + \zeta_2\beta_1$. As ζ_2, β_1 can be chosen arbitrarily in $\mathbf{F}$, there are q^2 quadruples $(\zeta_1, \beta_1, \zeta_2, \beta_2)$ satisfying (3.14) and (3.15). Therefore there are q^2 such hyperbolic pairs (v_1, v_2). Assume that $\mu \neq 0$. If $\zeta_1 = 0$, then $\zeta_2\beta_1 = -1, \beta_1 = \lambda\mu^{-1}$. Hence

$$\begin{cases} \zeta_1 & = & 0 \\ \beta_2 & \in & \mathbf{F} \\ \zeta_2 & = & -\mu\lambda^{-1} \\ \beta_1 & = & \lambda\mu^{-1} \end{cases} \tag{3.16}$$

Thus there q solutions to (3.16). If $\zeta_1 \neq 0$, then $\beta_1 \neq \lambda\mu^{-1}$, $\zeta_1 = 1 - (\lambda^{-1}\mu)\beta_1$ and $\beta_2 = \zeta_1^{-1}(1 + \zeta_2\beta_1)$. Thus equations (3.14), (3.15) become

$$\begin{cases} \beta_1 & \in & \mathbf{F} \setminus \{\lambda\mu^{-1}\} \\ \zeta_2 & \in & \mathbf{F} \\ \beta_2 & = & (1 - \lambda^{-1}\mu\beta_1)^{-1}(1 + \zeta_2\beta_1) \\ \zeta_1 & = & 1 - \mu\lambda^{-1}\beta_1. \end{cases} \tag{3.17}$$

Since $\beta_1 \neq \lambda\mu^{-1}$ and $\beta_2 \in \mathbf{F}$, there are $(q-1)q$ solutions to (3.17). Therefore, there are

$q + q(q-1) = q^2$ such hyperbolic pairs (v_1, v_2). The proof is now completed. ■

Let $(V_i, \mathbf{F}_q, \mathbf{f_i}), i = 1, 2$ be classical symplectic geometries with $dimV_i = 2m_i$. Assume $(V, \mathbf{f}) = (V_1, \mathbf{f}_1) \otimes (V_2, \mathbf{f}_2)$ be a tensor decomposition of V with $m_1 < m_2$. Let $\beta_1 = \{a_1, b_1, \ldots, a_{m_1}, b_{m_1}\}, \beta_2 = \{e_1, f_1, \ldots, e_{m_2}, f_{m_2}\}$ be standard bases for $(V_1, \mathbf{f}_1), (V_2, \mathbf{f}_2)$, respectively, and $I_i = I(V_i, \mathbf{f}_i)$. Then $\beta = \beta_1 \otimes \beta_2$ is a basis for $V = V_1 \otimes V_2$. Let $x = a_1 \otimes e_1 + \xi b_1 \otimes f_1 \in V$, with $\xi = \pm 1$.

Lemma 3.52 *Assume the notation above. Then*
(1) $|\langle x\rangle(I_1 \otimes I_2)| = \frac{1}{(q-1)(q^2-1)} q^{2m_1+2m_2-3}(q^{2m_1}-1)(q^{2m_2}-1)$;
(2) $|\{\langle v\rangle \in \langle x\rangle(I_1 \otimes I_2) \mid (v, x) \neq 0\}| = q^{4m_1+4m_2-6}$.

Proof. (1) We first determine $(I_1 \otimes I_2)_x$. For any $g \in (I_1 \otimes I_2)_x$, there exist $g_i \in I_i$ such that $g = g_1 \otimes g_2$ and $xg = x$. Let $w_1 = e_1 g_2, w_2 = f_1 g_2$, and

$$\begin{cases} a_1 g_1 &= \zeta_1 a_1 + \beta_1 b_1 + u_1 \\ b_1 g_1 &= \zeta_2 a_1 + \beta_2 b_1 + u_2, \end{cases}$$

where $u_i \in U_1 = \langle a_1, b_1\rangle^{\perp}$ and $\zeta_i, \beta_i \in \mathbf{F}$ for $i = 1, 2$. Since $g_i \in I_i$, we have $\mathbf{f_2}(w_1, w_2) = \mathbf{f_2}(e_1 g_2, f_1 g_2) = \mathbf{f_2}(e_1, f_1) = 1$, and similarly $\mathbf{f_1}(a_1 g_1, b_1 g_1) = \mathbf{f_1}(a_1, b_1) = 1$. Now as $\mathbf{f_1}(a_1 g_1, b_1 g_1) = \zeta_1\beta_2 - \zeta_2\beta_1 + \mathbf{f_1}(u_1, u_2)$, it follows that

$$\zeta_1\beta_2 - \zeta_2\beta_1 + \mathbf{f_1}(u_1, u_2) = 1 \tag{3.18}$$

We have $xg = a_1 g_1 \otimes e_1 g_2 + \xi b_1 g_1 \otimes f_1 g_2 = (\zeta_1 a_1 + \beta_1 b_1 + u_1) \otimes w_1 + \xi(\zeta_2 a_1 + \beta_2 b_1 + u_2) \otimes w_2$, or $xg = a_1 \otimes (\zeta_1 w_1 + \xi\zeta_2 w_2) + b_1 \otimes (\beta_1 w_1 + \xi\beta_2 w_2) + u_1 \otimes w_1 + u_2 \otimes (\xi w_2)$. We deduce from $xg = x$ that $a_1 \otimes (\zeta_1 w_1 + \xi\zeta_2 w_2) + b_1 \otimes (\beta_1 w_1 + \xi\beta_2 w_2) + u_1 \otimes w_1 + \xi u_2 \otimes w_2 = a_1 \otimes e_1 + \xi b_1 \otimes f_1$, and so $a_1 \otimes (\zeta_1 w_1 + \xi\zeta_2 w_2 - e_1) + b_1 \otimes (\beta_1 w_1 + \xi\beta_2 w_2 - f_1) + u_1 \otimes w_1 + \xi u_2 \otimes w_2 = 0$. As $\{a_1, b_1\}$ is linearly independent and $u_i \in \langle a_1, b_1\rangle^{\perp}$, $\zeta_1 w_1 + \xi\zeta_2 w_2 - e_1 = 0 = \beta_1 w_1 + \xi\beta_2 w_2 - f_1$, and $u_1 \otimes w_1 + \xi u_2 \otimes w_2 = 0$. Furthermore $\{w_1, w_2\}$ is also linearly independent, hence $u_1 = 0 = u_2$. In summary, we have

$$\begin{cases} \zeta_1 w_1 + \xi \zeta_2 w_2 &= e_1 \\ \beta_1 w_1 + \xi \beta_2 w_2 &= f_2 \\ u_1 &= 0 \\ u_2 &= 0. \end{cases} \tag{3.19}$$

As $u_1 = u_2 = 0$, it follows from (3.18) that $\zeta_1\beta_2 - \zeta_2\beta_1 = 1$. Using this together with the first two equations in (3.19), we get

$$\begin{cases} w_1 = e_1 g_2 = \beta_2 e_1 - \xi\zeta_2 f_1 \\ w_2 = f_1 g_2 = -\beta_1 e_1 + \zeta_1 f_1. \end{cases}$$

We conclude that $g_1 \in Sp(\langle a_1, b_1\rangle) \times Sp(U_1), g_2 \in Sp(\langle e_1, f_1\rangle) \times Sp(U_2)$, where $U_1 = \langle a_1, b_1\rangle^\perp$, $U_2 = \langle e_1, f_1\rangle^\perp$. Let $B_1 = \{a_1, b_1\}$, and $B_2 = \{e_1, f_1\}$. Then for any $\zeta_i, \beta_i \in \mathbf{F}, i = 1, 2$, with $\zeta_1\beta_2 - \zeta_2\beta_1 = 1$, set

$$A_1 = \begin{pmatrix} \zeta_1 & \beta_1 \\ \zeta_2 & \beta_2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \beta_2 & -\xi\zeta_2 \\ -\xi\beta_1 & \zeta_1 \end{pmatrix}, \tag{3.20}$$

and define T to be a subgroup of $I_1 \otimes I_2$ consisting of all elements $g_1 \otimes g_2$, where $(g_i)_{B_i} = A_i$ as above and g_i act trivially on U_i. Then $T \cong Sp_2(3) \cong SL_2(3)$. Hence

$$(I_1 \otimes I_2)_x = (Sp(U_1) \otimes Sp(U_2)).T,$$

where $Sp(U_i) \cong Sp_{2m_i-2}(3)$. Therefore

$$|x(I_1 \otimes I_2)| = |(I_1 \otimes I_2) : (I_1 \otimes I_2)_x| = \frac{1}{q^2-1} q^{2m_1+2m_2-3}(q^{2m_1}-1)(q^{2m_2}-1),$$

and hence (1) follows. (2) For $g = g_1 \otimes g_2 \in I_1 \otimes I_2$, write $u_1 = a_1g_1, u_2 = b_1g_1, w_1 = e_1g_2, w_2 = f_1g_2$, then $xg = u_1 \otimes w_1 + \xi u_2 \otimes w_2$ and $(u_1, u_2), (w_1, w_2)$ are hyperbolic pairs in V_1, V_2 respectively. As $Sp(V_i)$ act transitively on the set of all hyperbolic pairs $\mathcal{H}(V_i)$ by Proposition 3.3 in [14], it follows that $x(I_1 \otimes I_2)$ consists of all elements of the form $u_1 \otimes w_1 + \xi u_2 \otimes w_2$, where $(u_1, u_2) \in \mathcal{H}(V_1)$ and $(w_1, w_2) \in \mathcal{H}(V_2)$. However when a couple of hyperbolic pairs $((u_1, u_2), (w_1, w_2)) \in \mathcal{H}(V_1) \times \mathcal{H}(V_2)$ are fixed, there are $q(q^2 - 1)$ pairs of hyperbolic pairs $((u_1A_1, u_2A_1), (w_1A_2, w_2A_2)) \in \mathcal{H}(V_1) \times \mathcal{H}(V_2)$ which give rise to the same element $u_1 \otimes w_1 + \xi u_2 \otimes w_2$, where $A_i, i = 1, 2$ are as in (3.20). Thus

$$x(I_1 \otimes I_2) = \{u_1 \otimes w_1 + \xi u_2 \otimes w_2 \mid (u_1, u_2) \in \mathcal{HP}_*(V_1), (w_1, w_2) \in \mathcal{H}(V_2)\} \tag{3.21}$$

where $\mathcal{HP}_*(V_1)$ is define to be the set of of hyperbolic pairs which generates distinct hyperbolic planes in V_1. Let $v = u_1 \otimes w_1 + \xi u_2 \otimes w_2$ be such that $\mathbf{f}(v, x) = \lambda \neq 0$, where $(u_1, u_2) \in \mathcal{HP}_*(V_1)$ and $(w_1, w_2) \in \mathcal{H}(V_2)$. Then

$$\mathbf{f_1}(u_1, a_1)\mathbf{f_2}(w_1, e_1) + \xi\mathbf{f_1}(u_1, b_1)\mathbf{f_2}(w_1, f_1) + \xi\mathbf{f_1}(u_2, a_1)\mathbf{f_2}(w_2, e_1) + \mathbf{f_1}(u_2, b_1)\mathbf{f_2}(w_2, f_1) = \lambda.$$

Since λ is non-zero, one of the terms in the left side must be non-zero. Without loss, we assume that $\mathbf{f_1}(u_1, a_1)\mathbf{f_2}(w_1, e_1) \neq 0$. It follows that

$$\mathbf{f_2}(w_1, e_1) = \mathbf{f_1}(u_1, a_1)^{-1}(\lambda - \xi\mathbf{f_1}(u_1, b_1)\mathbf{f_2}(w_1, f_1) - \xi\mathbf{f_1}(u_2, a_1)\mathbf{f_2}(w_2, e_1) - \mathbf{f_1}(u_2, b_1)\mathbf{f_2}(w_2, f_1)).$$

By Lemma 3.51(4), there are q^{4m_1-4} hyperbolic planes with one component fixed in V_1, and by (3) of the same Lemma, there are q^{4m_2-2} hyperbolic pairs with one component fixed in V_2. Thus $|\{v \in x(I_1 \otimes I_2) \mid (v, x) \neq 0\}| = (q-1).q^{4m_1-4}.q^{4m_2-2} = (q-1)q^{4m_1+4m_2-6}$. ■

Proposition 3.53 *Assume G is nearly simple primitive rank 3 of type $\Omega^+_{2m}(3)$ and M is of type $Sp_{2m_1}(3) \otimes Sp_{2m_2}(3)$, with $m_1 < m_2$. There is an M-orbit on $\mathfrak{E}^{\varepsilon}_{\xi}(V)$ such that* *equation (3.1) does not hold unless $m_1 = 1$, in which case M has only one orbit on $\mathfrak{E}^+_\xi(V)$ so that $1_P^G \not\leqslant 1_M^G$ and hence M is in Table 1.1.*

Proof. Retain the notations before Lemma 3.52. Set $\delta_i = \delta_{\beta_i,\mathbf{f}_i}(-1)$, and $z = \delta_1 \otimes \delta_2^{-1}$. It follows from Lemma 4.4.5 and Proposition 4.4.12 in [29] that $z \in S$, $z^2 \in I_1 \otimes I_2$, $z \notin I_1 \otimes I_2$ and

$$M_\Omega = \begin{cases} I_1 \otimes I_2 & \text{if } n \equiv 4 \pmod 8 \\ (I_1 \otimes I_2)\langle z\rangle & \text{if } n \equiv 0 \pmod 8). \end{cases}$$

Since $x = a_1 \otimes e_1 + \xi b_1 \otimes f_1 \in V$, with $\xi = \pm 1$ and $\mathbf{f} = \mathbf{f_1} \otimes \mathbf{f_2}$, $(x,x) = \mathbf{f}(x,x) = \xi\mathbf{f}(a_1\otimes e_1, b_1\otimes f_1)+\xi\mathbf{f}(b_1\otimes f_1, a_1\otimes e_1) = \xi\mathbf{f_1}(a_1,b_1)\mathbf{f_2}(e_1,f_1)+\xi\mathbf{f_1}(b_1,a_1)\mathbf{f_2}(f_1,e_1) = 2\xi \neq 0$, so that x is non-singular in V. As $I_1\otimes I_2 \leqslant M_\Omega \leqslant M_I = (I_1\otimes I_2)\langle z\rangle$, and z fixes x, it follows that $x(I_1 \otimes I_2) = x(I_1 \otimes I_2)\langle z\rangle$. Thus $xM_\Omega = x(I_1 \otimes I_2)$. Therefore by Lemma 3.52(1), $|\langle x\rangle M_\Omega| = \frac{1}{(q-1)(q^2-1)}q^{2m_1+2m_2-3}(q^{2m_1}-1)(q^{2m_2}-1)$, and so $1+c+d = \frac{1}{16}3^{2m_1+2m_2-3}(3^{2m_1}-1)(3^{2m_2}-1)$. We first assume that $m_1 \geqslant 3$. As $16 > 3^2$ and $3^{2m_i}-1 < 3^{2m_i}$, we have $1+c+d < 3^{4m_1+4m_2-5}$. We will show that $3^{4m_1+4m_2-5} < 3^{m-1}$ so that $1+c+d < 3^{m-1}$ and so in view of (3.10), equation (3.1) cannot hold. As $m = 2m_1m_2$, $3^{4m_1+4m_2-5} < 3^{m-1}$ is equivalent to $(m_1-2)(m_2-2) > 0$. This is true because $m_2 > m_1 \geqslant 3$. Therefore we only need to consider cases $1 \leqslant m_1 \leqslant 2$. If $m_1 = 1$, then $m = 2m_2$ and $1+c+d = \frac{1}{2}3^{m-1}(3^m-1)$. By Lemma 3.35, $1+c+d = |\mathfrak{E}^+(V)|$, hence M_Ω has only one orbit on $\mathfrak{E}^+(V)$. If $m_1 = 2$, then $m = 4m_2$ and by Lemma 3.52(2), we have $1+c = |\{\langle v\rangle \in \langle x\rangle(I_1\otimes I_2) \mid (v,x) \neq 0\}| = 3^{4m_1+4m_2-6}$ and so $d = 2.3^{4m_2+1} - 5.3^{2m_2+1}$, $c - 2d = 10.3^{2m_2+1} - 3^{4m_2+1} - 1$. Assume that (3.8) holds. Then $c - 2d = 3^{m-1} - 1$, and so $10.3^{2m_2+1} - 3^{4m_2+1} - 1 = 3^{4m_2-1} - 1$. It follows that $3^{2m_2-2} = 1$, hence $m_2 = 1$, a contradiction to our assumption $m_2 > m_1 = 2$. Assume that (3.9) holds. Then $(3^{m-1}+1)(3^m - 3 + c - 2d) = 4c$ or equivalently, $3^{2m-1} - 3 + (3^{m-1}-3)c = (2.3^{m-1}+2)d$. Since $A := 1+c+d = 5.3^{2m_2+1}(3^{2m_2}-1)$, we have $c = A - 1 - d$, and hence $3^{2m-1} - 3 + (3^{m-1}-3)(A-d-1) = (2.3^{m-1}+2)d$, or $3^{m-1}(3^m-1) + 3A(3^{m-2}-1) = (3^m-1)d$. It follows that $3A(3^{m-2}-1)$ is divisible by

$3^m - 1$, where $m = 4m_2$. However as $3A(3^{m-2} - 1) = \frac{1}{16}(3^4 - 1).3^{2m_2+2}(3^{2m_2} - 1)(3^{m-2} - 1)$ and $m - 2 < m, 4 < m$ and $2m_2 < m$, by Zsigmondy's Theorem $3^m - 1$ has a primitive prime divisor which does not divide $3A(3^{m-2} - 1)$. Thus (3.9) cannot hold. Therefore (3.1) cannot hold when $m_1 = 2$. ■

The Symplectic-type normalizers $\mathcal{C}_6$

Let α be a prime. Recall that an α-group R is *extraspecial* if $Z(R) = \Phi(R) = R' \cong \mathbb{Z}_\alpha$. Also, R is of *symplectic-type* if every characteristic abelian subgroup of R is cyclic. Assume that $\alpha = 2$. Up to isomorphism, there are only two extraspecial groups of order 2^3, namely, Q_8 and D_8. Then any extraspecial group R is either a central product of ℓ copies of D_8, or $\ell - 1$ copies of D_8 and one Q_8. The first type is denoted by $2_+^{1+2\ell}$ and the latter one $2_-^{1+2\ell}$. Suppose that $R \cong 2_+^{1+2\ell}$. Then $C_{Aut(R)}(Z(R)) \cong 2^{2\ell}.O_{2\ell}^+(2)$, and R is of symplectic-type with exponent 4. By Proposition 4.6.3[29], R has exactly one faithful absolutely irreducible representation ρ_1 over an algebraically closed field of characteristic $p \neq 2 = \alpha$. Moreover, this representation has degree 2^ℓ and leaves invariant a non-degenerate quadratic form. Thus the members of $\mathcal{C}_6(\Xi)$ are groups $M_\Xi = N_\Xi(R)$, where $R \cong 2_+^{1+2\ell}$. By (4.6.1)[29], $\overline{M}_\Xi \cong C_{Aut(R)}(Z(R)) \cong 2^{2\ell}.O_{2\ell}^+(2)$.

Proposition 3.54 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^+(3)$ and M is of type $2_+^{1+2\ell}.O_{2\ell}^+(2)$, with $\ell \geqslant 3$. There is an M-orbit on $\mathfrak{E}_\xi^\varepsilon(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. By Proposition 4.6.8[29], $\overline{M}_\Omega = M_{\overline{\Omega}} \cong 2^{2\ell}.\Omega_{2\ell}^+(2)$ and $M_{\overline{I}} \leqslant \overline{M}_\Xi \cong 2^{2\ell}.O_{2\ell}^+(2)$. Let x_ξ be any non-singular vector in V. Since $|\langle x_\xi\rangle\overline{M}| = |\overline{M} : \overline{M}_{\langle x_\xi\rangle}| \leqslant |M_{\overline{I}}|$, it follows that $A = 1 + c + d \leqslant 2^{2\ell}|O_{2\ell}^+(2)| < 2^{2\ell^2+\ell+1}$. In view of (3.10), it suffices to show that $2^{2\ell^2+\ell+1} < 3^{n/2-1}$. Observe that this is true if $\ell \geqslant 8$. If $\ell = 3$, then $2m = 2^3 = 8$. By [9], $\overline{M}_\Omega$ is not maximal in $P\Omega_8^+(3)$. Therefore, we assume that $4 \leqslant \ell \leqslant 7$. Let $R \cong 2_+^{1+2\ell}$. We now describe the faithful irreducible ρ_1 of R. As in discussion previous Proposition 4.6.8[29], for $j = 1, \ldots, \ell$, let $(V_j, \mathbf{F}, \mathbf{f}_j)$ be a 2-dimensional orthogonal geometry with $D(\mathbf{f}_j) = \square$. Let $\beta_j = \{v_{j,1}, v_{j,2}\}$ be an orthonormal basis of V_j and set $x_j, y_j \in I(V_j, \mathbf{F}, \mathbf{f}_j)$ satisfy

$$(x_j)_{\beta_j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } (y_j)_{\beta_j} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $R_j := \langle x_j, y_j \rangle \cong D_8$ and

$$R \cong 2_+^{1+2\ell} \cong R_1 \circ \cdots \circ R_\ell \leqslant I(V_1 \otimes \cdots \otimes V_\ell, \mathbf{F}, \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_\ell) \cong O_{2\ell}^+(3).$$

Let $\overline{R} = R/Z(R)$. Then $\overline{R}$ is a 2ℓ-dimensional vector space over $\mathbf{F}_2$ and $\overline{R}$ admits a non-degenerate quadratic form P such that $P(\overline{x}) = 0$ or 1 according as $|x| = 2$ or 4. Also, $\mathbf{f}_P(\overline{x}, \overline{y}) = [x, y]$ for any $x, y \in R$. Observe that if E is a subgroup of R, then $\overline{E}$ is a totally singular subspace of $\overline{R}$ if and only if E is an elementary abelian subgroup. It follows that $(\overline{R}, \mathbf{F}_2, P)$ is a classical orthogonal geometry of Witt index 0, so that $I(\overline{R}, \mathbf{F}_2, P) \cong O_{2\ell}^+(2)$. Identify R with its image $R\rho_1$ in $\Omega(V)$ and set $E_\xi = \langle y_\xi, y_2, \ldots, y_\ell \rangle$, where $\xi = \pm$ and $y_+ = y_1, y_- = x_1^{-1} y_1$. As $|y_i| = 2 = |y_\xi|$ and $\{y_\xi, y_2, \ldots, y_\ell\}$ are pair-wise commuting so that E_ξ are elementary abelian subgroups of R, hence $\overline{E}_\xi$ are maximal totally singular subspace of $\overline{R}$. Let $v_+ = v_{1,1}$ and $v_- = v_{1,1} + v_{1,2}$ and $x_\xi = v_\xi \otimes v_{2,1} \otimes \cdots \otimes v_{\ell,1} \in V$, where $V = V_1 \otimes \cdots \otimes V_\ell$. Observe that $v_\xi y_\xi = v_\xi$ and $v_{j,1} y_\xi = v_{j,1}$ for any $j \geqslant 2$, and this is also true for all y_j with $j \geqslant 2$, hence

$$x_\xi E_\xi = x_\xi. \tag{3.22}$$

Let $A = Aut(R), K = C_A(Z(R)) \cong \overline{M}_\Xi$ and $A^* = Out(R)$. By Exercise 8.5, p.116 in [1], we have $Inn(R) = C_A(\overline{R})$ and $A^* \cong O_{2\ell}^+(2)$. As $Inn(E_\xi)$ centralizes E_ξ, $E_{2^\ell} \cong Inn(E_\xi) \leqslant C_K(E_\xi)$, and in $(\overline{R}, \mathbf{F}_2, P)$, $\overline{E}_\xi$ is a maximal totally singular subspace, hence if $\overline{\alpha} \in N_{A^*}(\overline{E}_\xi)$ then $\overline{\alpha}$ leaves $\overline{E}_\xi$ invariant. For each $\overline{\alpha} \in N_{A^*}(\overline{E}_\xi)$, there exist an element $\beta \in A$ so that $\overline{\beta} = \overline{\alpha}$ and $E_\xi \beta = E_\xi$. In fact, fix a representative $\alpha \in A$ of

$\overline{\alpha}$, define $x_i\beta = x_i\alpha$ and $y_i\beta = y_i'$, where $y_i\alpha = z^{j_i}y_i', y_i' \in E_\xi, j_i = 0, 1, i = 1, \ldots, \ell$ and $\langle z \rangle = Z(R)$. Then β is the required element. Since $N_{\Omega_{2\ell}^+(2)}(\overline{E}_\xi) \cong 2^{(\ell^2-\ell)/2}.GL_\ell(2)$, and $\overline{M}_\Omega \cong Inn(R).\Omega_{2\ell}^+(2)$, it follows that $|\overline{M}_{\langle x_\xi \rangle}| \geqslant |Inn(E_\xi)|.|N_{\Omega_{2\ell}^+(2)}(\overline{E}_\xi)|$, and hence $|\langle x_\xi \rangle M| \leqslant |\overline{M} : \overline{M}_{\langle x_\xi \rangle}| \leqslant 2^\ell \prod_{i=1}^{\ell-1}(2^i+1)$. Since $4 \leqslant \ell \leqslant 7$, we have $2^\ell \prod_{i=1}^{\ell-1}(2^i+1) < 3^{2^{\ell-1}-1}$. Thus (3.1) cannot hold. ∎

The tensor product subgroups $\mathcal{C}_7$

Let V_1 be an α-dimensional vector space over $\mathbf{F}$, with a non-degenerate bilinear form $\mathbf{f}_1$. Let $(V_i, \mathbf{f}_i), i = 1, \ldots, b$ be classical geometries which are similar to $(V_1, \mathbf{f}_1)$. Then for each i, there exists a similarity $\eta_i : (V_i, \mathbf{f}_i) \mapsto (V_1, \mathbf{f}_1)$ satisfying $\mathbf{f}_i(v\eta_i, w\eta_i) = \lambda_i \mathbf{f}_1(v, w)$ for all $v, w \in V_1$, where $\lambda_i \in \mathbf{F}^*$ is independent of v, w. Let $\mathcal{D}$ be the tensor decomposition $(V, \mathbf{f}) = (V_1 \otimes \cdots \otimes V_b, \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_b)$. Then $\dim V = \alpha^b$ and V is spanned by the vectors $v_1\eta_1 \otimes \cdots \otimes v_b\eta_b$ with $v_i \in V_1$. Define $\alpha_i : \Xi_1 \mapsto \mathit{\Gamma L}(V_i)$ by $(v\eta_i)(g\alpha_i) = (vg)\eta_i \quad (g \in \Xi_1, v \in V_1)$. Then α_i is an isomorphism from Ξ_1 to Ξ_i. We will write $v_1 \otimes \cdots \otimes v_b$ and $g_1 \otimes \cdots \otimes g_b$ instead of $v_1\eta_1 \otimes \cdots \otimes v_b\eta_b$ and $g_1\alpha_1 \otimes \cdots \otimes g_b\alpha_b$. Now let $\beta \in S_b$ and define g_β by $(v_1 \otimes \cdots \otimes v_b)g_\beta = v_{1\beta^{-1}} \otimes \cdots \otimes v_{b\beta^{-1}}$. Then $S_b \cong J := \langle g_\beta \mid \beta \in S_b \rangle \leqslant I$. Let M_I be the tensor decomposition $\mathcal{D}$ in I. As described in §4.7,

$$M_I = (I_1 \otimes \cdots \otimes I_b)\langle z_{i,i+1} \mid i < b \rangle.J \tag{3.23}$$

where $z_{i,j} = \delta_i \otimes \delta_j^{-1}$ and $\delta_i = \delta_{\beta_i, \mathbf{f}_i}(\mu)$, with $\langle \mu \rangle = \mathbf{F}^*$ and δ_{β_i, f_i} is defined as in (2.4.3), (2.4.8) in [29]. Denote $X_\sharp = X_1 \otimes \cdots \otimes X_b$ for $X \in \{I, S, \Omega\}$.

Proposition 3.55 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^+(3)$ and M is of type $O_{2a}^\varepsilon(3) \wr S_b$, and $a \geqslant 3$ if $\varepsilon = +$ and $a \geqslant 2$ if $\varepsilon = -$. There is an M-orbit on $\mathfrak{E}_\xi^\varepsilon(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. Let V_1 be a $2a$-dimensional vector space over $\mathbf{F}$, with a non-degenerate symmetric bilinear form $\mathbf{f}_1$. Let $(V_i, \mathbf{f}_i), i = 1, \ldots, b$ be classical geometries which are similar to

$(V_1, \mathbf{f}_1)$ and $(V, \mathbf{f}) = (V_1 \otimes \cdots \otimes V_b, \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_b)$. Let $\varepsilon = sgnV_1$. We first assume that $b \geqslant 3$. Let $\beta = \{e_1, \ldots, e_a, f_1, \ldots, f_a\}$ be a standard basis for V_1. For $\xi = \pm 1$, put $x_\xi = e_1 \otimes \cdots \otimes e_1 + \xi f_1 \otimes \cdots \otimes f_1 \in V$. As $\mathbf{f} = \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_b$ and $\mathbf{f}_i = \lambda_i \mathbf{f}_1$, it follows that $\mathbf{f}(x, x) = 2\xi \prod_{i=2}^{b} \lambda_i \mathbf{f}_1(e_1, f_1)^{b-1} = 2\xi \prod_{i=2}^{b} \lambda_i \neq 0$, so that x_ξ is a non-singular vector in V. Now by Proposition 4.4.13$(iii)(a)$ in [29], $S_\sharp \leqslant \Omega$, and hence $S_\sharp \leqslant M_\Omega \leqslant M_I$. Let $C_i = C_{S_i}(\langle e_1, f_1 \rangle) \cong SO(\langle e_1, f_1 \rangle^\perp)$. Since $sgn\langle e_1, f_1 \rangle = +$, we have $sgn\langle e_1, f_1 \rangle^\perp = \varepsilon$ and so $C_i \cong SO^\varepsilon_{2a-2}(3)$. Set $N = C_1 \otimes \cdots \otimes C_b \leqslant M_\Omega$. Then $x_\xi N = x_\xi$. Further observe that J and $z_{i,i+1}, 1 \leqslant i < b$ also fix x_ξ. Thus $N\langle z_{i,i+1} | i < b \rangle.J \leqslant (M_\Omega)_{x_\xi}$. Therefore $|M_\Omega : (M_\Omega)_{x_\xi}| \leqslant |M_I : N\langle z_{i,i+1} | i < b\rangle.J| = |O^\varepsilon_{2a}(3) : SO^\varepsilon_{2a-2}(3)|^b$, and hence $|\langle x_\xi \rangle M_\Omega| < |O^\varepsilon_{2a}(3) : SO^\varepsilon_{2a-2}(3)|^b = [2.3^{2a-2}(3^a - \varepsilon)(3^{a-1} + \varepsilon)]^b$. As $(3^a - \varepsilon)(3^{a-1} + \varepsilon) < 2.3^{2a-1}$ and $4 < 3^2$, it follows that $|\langle x_\xi \rangle M_\Omega| < 3^{(4a-1)b}$. In view of (3.10), it suffices to show that $3^{(4a-1)b} < 3^{m-1}$, where $n = 2m = (2a)^b$. This is equivalent to $(4a-1)b \leqslant 2^{b-1}a^b - 1$. This is correct as $b \geqslant 3$ and $a \geqslant 2$. Hence equation (3.1) cannot hold. Now, assume that $b = 2$. Let $v_\xi = e_1 + \xi f_1, w = e_1 + f_1$ and $x_\xi = v_\xi \otimes w$. Then $\mathbf{f}(x_\xi, x_\xi) = \mathbf{f}_1(v_\xi, v_\xi)\mathbf{f}_2(w, w) = \xi\lambda_2 \neq 0$, and so x_ξ is non-singular in $V = V_1 \otimes V_2$. As above, we have $S_1 \otimes S_2 \leqslant M_\Omega$. Let $C_1 = C_{S_1}(v_\xi)$ and $C_2 = C_{S_2}(w)$. Since v_ξ, w are non-singular vectors in V_1 and V_2 respectively, we have $C_i \cong SO_{2a-1}(3)$. Set $N = C_1 \otimes C_2$. Then $x_\xi N = x_\xi$. Clearly, as $v_\xi I_1 = v_\xi S_1$ and $w I_2 = w S_2$, we have $x_\xi(I_1 \otimes I_2) = x_\xi(S_1 \otimes S_2)$. Thus $|x_\xi M_\Omega| = |x_\xi(S_1 \otimes S_2)\langle z_{1,2}\rangle J| \leqslant |S_1 : C_1|^2.2^2$ as $z_{1,2}^2 = 1$ and $J \cong \mathbb{Z}_2$. Therefore $|\langle x_\xi \rangle M_\Omega| \leqslant \frac{1}{2}|SO^\varepsilon_{2a}(3) : SO_{2a-1}(3)|^2.4 = 2.3^{2a-2}(3^a - \varepsilon)^2$. By (3.10), we need to prove that $2.3^{2a-2}(3^a - \varepsilon.1)^2 < 3^{m-1}$, where $n = 2m = (2a)^2$. As $2(3^a - \varepsilon.1)^2 \leqslant 2(3^a + 1)^2 < 3^{2a+1}$, The above inequality is equivalent to $3^{4a-1} \leqslant 3^{m-1}$ or $4a \leqslant 2a^2$. This is obviously true for any $a \geqslant 2$. Thus equation (3.1) cannot hold. ■

Proposition 3.56 *Assume G is nearly simple primitive rank 3 of type $\Omega^+_{2m}(3)$ and M is of type $Sp_{2a}(3) \wr S_b$ with b even and $a \geqslant 2$. There is an M-orbit on $\mathfrak{E}^\varepsilon_\xi(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. Let V_1 be a $2a$-dimensional vector space over $\mathbf{F}$, with non-degenerate symplectic bilinear form $\mathbf{f}_1$. Let $(V_i, \mathbf{f}_i), i = 1, \ldots, b$ be classical geometries which are similar to $(V_1, \mathbf{f}_1)$ and $(V, \mathbf{f}) = (V_1 \otimes \cdots \otimes V_b, \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_b)$. Then $\mathbf{f} = \mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_b$ and $\mathbf{f}_i = \lambda_i \mathbf{f}_1$, where $\lambda_i \in \mathbf{F}^*$. Let $\beta = \{e_1, \ldots, e_a, f_1, \ldots, f_a\}$ be a standard basis for V_1 and let $x_\xi = e_1 \otimes \cdots \otimes e_1 + \xi f_1 \otimes \cdots \otimes f_1$, where $\xi = \pm 1$. As I_1 is quasi-simple, $I_\sharp \leqslant \Omega$, and hence $I_\sharp \leqslant M_\Omega \leqslant M_I$. Let $C_i = C_{I_i}(\langle e_1, f_1 \rangle) \cong Sp(\langle e_1, f_1\rangle^\perp)$. Hence $C_i \cong Sp_{2a-2}(3)$. Let $N = C_1 \otimes \cdots \otimes C_b$. Then N, J and $z_{i,i+1}, 1 \leqslant i < b$ fix vector x_ξ, so that $N.\langle z_{i,i+1} \mid 1 \leqslant i < b\rangle.J \leqslant (M_\Omega)_{x_\xi}$.

We have $|\langle x_\xi \rangle M_I| \leqslant |M_I : N.\langle z_{i,i+1} \mid 1 \leqslant i < b\rangle.J| = |Sp_{2a}(3) : Sp_{2a-2}(3)|^b = (3^{2a-1}.(3^{2a}-1))^b \leqslant 3^{(4a-1)b}$. Assume that $b \geqslant 4$. Since $m = 2^{b-1}a^b$ and $a \geqslant 2, b \geqslant 4$, we have $3^{m-1} > 3^{(4a-1)b}$, and so by (3.10), equation (3.1) cannot happen. Next assume that $b = 2$ and $a \geqslant 4$. Then as $(4a-1)b = 8a-2 < 2a^2 - 1 = 2^{b-1}a^b - 1$, so that equation (3.1) cannot hold. Thus we can assume that $b = 2$ and $2 \leqslant a \leqslant 3$. Set $z = z_{1,2} = \delta_1 \otimes \delta_2^{-1}, g = g_{(12)} \in J$ and $x = \lambda_2^{-1} e_1 \otimes e_1 + \xi f_1 \otimes f_1$, where $\mathbf{f}_2 = \lambda_2 \mathbf{f}_1$. Then $\mathbf{f}(x,x) = 2\xi$, hence x is non-singular in V. We will show that $xM_\Omega = x(I_1 \otimes I_2)$. Write $x = e_1 \otimes (\lambda_2^{-1} e_1) + \xi f_1 \otimes f_1$. Then $\mathbf{f}_1(e_1, f_1) = 1$ and $\mathbf{f}_2(\lambda_2^{-1}e_1, f_1) = \lambda_2 \mathbf{f}_1(\lambda_2^{-1} e_1, f_1) = 1$ so that $(e_1, f_1), (\lambda_2^{-1}e_1, f_1)$ are hyperbolic pairs in $(V_1, \mathbf{f}_1), (V_2, \mathbf{f}_2)$, respectively. For any $g_1 \otimes g_2 \in I_1 \otimes I_2$, $xg_1 \otimes g_2 = e_1 g_1 \otimes \lambda_2^{-1} e_1 g_2 + \xi f_1 g_1 \otimes f_1 g_2$. Clearly (e_1g_1, f_1g_1) and $(\lambda_2^{-1}e_1g_2, f_1g_2)$ are hyperbolic pairs in V_1 and V_2. Argue as in Lemma 3.52, we have $x(I_1 \otimes I_2) = \{u_1 \otimes \lambda_2^{-1} w_1 + \xi u_2 \otimes w_2 \mid (u_1, u_2) \in \mathcal{HP}_*(V_1), (\lambda_2^{-1} w_1, w_2) \in \mathcal{H}(V_2)\}$, where $\mathcal{HP}_*(V_1)$ is defined to be the set of of hyperbolic pairs which generates distinct hyperbolic planes in V_1 and $\mathcal{HP}(V_2)$ is the set of all hyperbolic pairs in V_2. Observe that if (u_1, u_2) and $(\lambda_2^{-1}w_1, w_2)$ are hyperbolic pairs in V_1, V_2, respectively, and $v = u_1 \otimes \lambda_2^{-1} w_1 + \xi u_2 \otimes w_2$, then $vz = (-u_1\delta_1) \otimes (-\lambda_2^{-1} w_1 \delta_2^{-1}) + \xi u_2 \delta_1 \otimes w_2 \delta_2^{-1}$. Since $\mathbf{f}_1(u_1\delta_1, u_2\delta_1) = -\mathbf{f}_1(u_1, u_2)$ and $\mathbf{f}_2(\lambda_2^{-1} w_1 \delta_2^{-1}, w_2\delta_2^{-1}) = -\mathbf{f}_2(\lambda_2^{-1} w_1, w_2)$, it follows that $\mathbf{f}_1(-u_1\delta_1, u_2\delta_1 = 1$ and $\mathbf{f}_2(-\lambda_2^{-1} w_1 \delta_2^{-1}, w_2 \delta_2^{-1}) = 1$. Hence $(-u_1\delta_1, u_2\delta_1), (-\lambda_2^{-1}w_1\delta_2^{-1}, w_2\delta_2^{-1})$ are still hyperbolic pairs in V_1, V_2. Thus $x(I_1 \otimes I_2)\langle z \rangle = x(I_1 \otimes I_2)$. Next, we have $vg = (u_1 \otimes \lambda_2^{-1} w_1)g + \xi(u_2 \otimes w_2)g = \lambda_2^{-1} w_1 \otimes u_1 + \xi w_2 \otimes u_2 =$

$w_1 \otimes \lambda_2^{-1}u_1 + \xi w_2 \otimes u_2$. Since $\mathbf{f}_2(\lambda_2^{-1}w_1, w_2) = 1$, $\lambda_2 = \mathbf{f}_2(w_1, w_2) = \lambda_2\mathbf{f}_1(w_1, w_2)$, hence $\mathbf{f}_1(w_1, w_2) = 1$. Similarly $\mathbf{f}_2(\lambda_2^{-1}u_1, u_2) = \lambda_2\mathbf{f}_1(\lambda_2^{-1}u_1, u_2) = \mathbf{f}_1(u_1, u_2) = 1$. Thus $vg \in x(I_1 \otimes I_2)$. Therefore $x(I_1 \otimes I_2) = xM_I$, and so, as $I_1 \otimes I_2 \leqslant M_\Omega \leqslant M_I$, it follows that $xM_\Omega = x(I_1 \otimes I_2)$. By Lemma 3.52, we have $1+c+d = \frac{1}{16}3^{4a-3}(3^{2a}-1)^2$, $c = 3^{8a-6}-1$, $d = 3^{4a-3}(\frac{1}{16}(3^{2a}-1)^2 - 3^{4a-3})$ and hence $c - 2d = 3^{8a-5} - 1 - \frac{1}{8}.3^{4a-3}(3^{2a}-1)^2$. Assume that equation (3.8) holds. Then $c-2d = 3^{m-1}-1$. Hence $3^{8a-5}-1-\frac{1}{8}.3^{4a-3}(3^{2a}-1)^2 = 3^{2a^2-1}-1$. As $4a-3 \leqslant 8a-5$ and $4a-3 \leqslant 2a^2-1$, it follows that $8(3^{4a-2} - 3^{2(a-1)^2}) = (3^{2a}-1)^2$. We have $2(a-1)^2 - (4a-2) = 2(a-2)^2 - 4$, and so, if $a \geqslant 4$, then $2(a-1)^2 - (4a-2) > 0$, so that $8(3^{4a-2} - 3^{2(a-1)^2}) < 0 < (3^{2a}-1)^2$. Now if $2 \leqslant a \leqslant 3$, then $4a-2 > 2(a-1)^2$, hence $8(3^{4a-2} - 3^{2(a-1)^2}) = 8.3^{2(a-1)^2}(3^{8a-4-2a^2} - 1) = (3^{2a}-1)^2$. It follows that $2(a-1)^2 = 0$. However this cannot happen as $a \geqslant 2$. Thus equation (3.8) cannot hold. Finally assume that equation (3.9) holds. Then $(3^m - 1)d = 3^{m-1}(3^m-1) + \frac{1}{16}3^{4a-2}(3^{m-2}-1)(3^{2a}-1)^2$ or equivalently, $16(3^m-1)(d - 3^{m-1}) = 3^{4a-2}(3^{m-2}-1)(3^{2a}-1)^2$. Since $m > m-2$ and $m = 2a^2 > 2a$, this equality cannot hold by Zsigmondy's Theorem. Thus equation (3.1) cannot hold. ■

3.5.3 Permutation Characters of Maximal subgroups in $\mathcal{S}$

Embedding of Symmetric and Alternating groups

Proposition 3.57 *Assume G is nearly simple primitive rank 3 of type $\Omega^{\varepsilon}_{2m}(3)$ and M is almost simple of type A_n, with $n \geqslant 10$, and V is the fully deleted permutation module for A_n in characteristic $p = 3$. There is an M-orbit on $\mathfrak{E}^{\varepsilon}_{\xi}(V)$ such that equation (3.1) does not hold unless $n = 5, 6, 7, 12$ or 18. If $n = 5, 6, 7$, then $\varepsilon = -$ and M has at most 2 orbits on $\mathfrak{E}^{-}_{\xi}(V)$ so that $1_P^G \nleqslant 1_M^G$ by Corollary 3.7; if $n = 12$, then $\xi = \square, \varepsilon = +$ and M has two orbits on $\mathfrak{E}^{+}_{\xi}(V)$, so that $1_P^G \nleqslant 1_M^G$ by Corollary 3.7; if $n = 18$ then $\xi = \square, \varepsilon = -$, M has 4 orbits on $\mathfrak{E}^{-}_{\xi}(V)$ and equation (3.1) holds for $r = s$ and for any M-orbits on $\mathfrak{E}^{-}_{\xi}(V)$. Thus these cases appear in Table 1.2.*

Proof. Assume the construction and notation before Lemma 3.23. We have

$$(e_i, e_j) = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| \geqslant 2. \end{cases} \quad \text{and } f_\beta = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

so that $D(f_\beta) = n - \varepsilon_3(n)$. Let $v = \varepsilon_1 - \varepsilon_2,$ or $\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \in V$. Let Q be the quadratic form associated to the natural bilinear form f on $\mathbf{F}_3^n$, we have $Q(v) \neq 0$, hence v is a non-singular vector in V with $\xi = Q(v) = \boxtimes, \square$, respectively. We see that $n - 1 - \varepsilon_3(n)$ is even if and only if $n = 6k - 1, n = 6k$ or $n = 6k + 1$, where $k \geqslant 1$.

(a) Parameters for $v = \varepsilon_1 - \varepsilon_2$. We have $\xi = Q(v) = 2$, and by Lemma 3.23,

$$\begin{cases} 1 + c + d &= \frac{1}{2}n(n-1) \\ c &= 2n - 4 \\ d &= \frac{1}{2}(n-2)(n-3). \end{cases}$$

As $2m = n - 1 - \varepsilon_3(n) \geqslant n - 2$, we have $3^{m-2} \geqslant 3^{(n-6)/2}$. We see that if $n \geqslant 15$ then $3^{(n-6)/2} > \frac{1}{2}n(n-1)$, and hence $3^{m-2} > 1 + c + d$. In view of (3.10) and (3.11), equation (3.1) cannot hold. Thus we can assume that $5 \leqslant n \leqslant 14$. However as n has the form $6k - 1, 6k$ or $6k + 1$, it follows that $n \in \{5, 6, 7, 11, 12, 13\}$.

(1) Case $n = 5$. Then $2m = 5 - 1 - \varepsilon_3(5) = 4$, $D = D(f_\beta) = 5 \equiv -1 = \boxtimes$, hence $\varepsilon = sgnV = (-)^{m-1} = -$. Thus A_5 can embed in $\Omega_4^-(3) \cong A_6$. Let $z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5$. Then $z \in V, Q(z) = Q(v)$ and $|\langle z \rangle M| = 5$. As $|\mathfrak{E}_2^-(V)| = \frac{1}{2}3^{m-1}(3^m - \varepsilon.1) = 15 = 10 + 5$, it follows that M has only 2 orbits on $\mathfrak{E}_2^-(V)$.

(2) Case $n = 6$. Then $2m = 6 - 1 - \varepsilon_3(6) = 4$, $D = 5 \equiv -1 = \boxtimes$, hence $\varepsilon = sgnV = (-)^{m-1} = -$. Thus A_6 can embed in $\Omega_4^-(3) \cong A_6$. Hence M has one orbit on $\mathfrak{E}_2^-(V)$.

(3) Case $n = 7$. Then $2m = 7-1-\varepsilon_3(7) = 6$, and so $m = 3$. Also $D = D(f_\beta) = 7 \equiv 1 = \square$, hence $\varepsilon = sgnV = (-)^m = -$. Thus A_7 can embed in $\Omega_6^-(3) \cong U_4(3)$. In this case, there are only two orbits with representatives v and $z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5$, and $|\langle z\rangle M| = \dfrac{7!}{1!4!(7-5)!} = 105$.

(4) Case $n = 11$. Then $2m = 11-1-\varepsilon_3(11) = 10$, $D = 11 \equiv -1 = \boxtimes$, hence $\varepsilon = sgnV = (-)^{m-1} = +$. Thus A_{11} can embed in $\Omega_{10}^+(3)$. Now $3^{m-1} = 3^4 > 55 = \frac{1}{2}n(n-1)$, in view of (3.10), equation (3.1) cannot hold.

(5) Case $n = 12$. Then $2m = 12-1-\varepsilon_3(12) = 10$, $D = D(f_\beta) = 11 \equiv -1 = \boxtimes$, hence $\varepsilon = sgnV = (-)^{m-1} = +$. Thus A_{12} can embed in $\Omega_{10}^+(3)$. As $3^{m-1} = 3^4 = 81 > 66 = \frac{1}{2}n(n-1)$, in view of (3.10), equation (3.1) cannot hold.

(6) Case $n = 13$. Then $2m = 13-1-\varepsilon_3(13) = 12$, $D = D(f_\beta) = 13 \equiv 1 = \square$, hence $\varepsilon = sgnV = (-)^m = +$. Thus A_{13} can embed in $\Omega_{12}^+(3)$. Now $3^{m-1} = 3^5 = 243 > 78 = \frac{1}{2}n(n-1)$, by (3.10), equation (3.1) cannot hold.

(b) Parameters for $v = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$. We have $\xi = Q(v) = 1$, and by Lemma 3.23

$$\begin{cases} 1+c+d &= \frac{1}{8}n(n-1)(n-2)(n-3) \\ c &= 2n^3 - 25n^2 + 111n - 172 \\ d &= 2 + 4(n-4)^2 + \frac{1}{8}(n-4)(n-5)(n-6)(n-7). \end{cases}$$

If $n \geqslant 26$, then $3^{(n-6)/2} > \frac{1}{8}n(n-1)(n-2)(n-3)$. Thus $3^{m-2} \geqslant 3^{(n-6)/2} > 1+c+d$. Therefore, (3.1) cannot happen. So we assume that $5 \leqslant n \leqslant 25$. As $n = 6k-1, 6k$ or $6k+1$, it follows that $n \in \{5, 6, 7, 11, 12, 13, 17, 18, 19, 23, 24, 25\}$. If $n = 5$, then $A_5 \leqslant \Omega_4^-(3) \cong A_6$, and $|\langle w\rangle M| = \frac{1}{8}5.4.3.2 = 15 = |\mathfrak{E}_1^-(V)|$. Thus there is only one orbit. As in previous case, A_6 has only one orbit on $\mathfrak{E}_1^-(V)$. If $n = 7$, then $A_7 \leqslant \Omega_6^-(3)$. Let $z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \in V$. Then $(z, z) = (v, v)$ and since there is no $g \in S_7$ which maps z to $-z$, $|\langle z\rangle M| = |zM| = \dfrac{7!}{5!2!} = 21$. Now, $|\mathfrak{E}_1^-(V)| = \frac{1}{2}3^{m-1}(3^m + \varepsilon.1) = 126 = 105 + 21 = |\langle w\rangle M| + |\langle z\rangle M|$. Hence M has two orbits on $\mathfrak{E}_1^-(V)$. Next suppose that $n = 6k-1, 6k$ or $6k+1$, where $2 \leqslant k \leqslant 4$.

(1) Subcase $n = 6k+1$. Let $z = \varepsilon_1 + \cdots + \varepsilon_{6k-3} + \varepsilon_{6k-2} + \varepsilon_{6k-1} - \varepsilon_{6k} - \varepsilon_{6k+1}$. Then $z \in V, (z,z) = (v,v)$ and $|\langle z\rangle M| = |zM| = \dfrac{n!}{(n-2)!2!} = \frac{1}{2}n(n-1)$. If $k \geqslant 3$ then $n \geqslant 19$, hence $3^{m-2} \geqslant 3^{(n-6)/2} > \frac{1}{2}n(n-1) = |zM|$. If $k = 2$ then $n = 13$, so that $\varepsilon = sgnV = +$. In this case $2m = 12$, and so $3^{m-1} = 3^5 > 78 = |\langle z\rangle M|$. In view of (3.10), (3.1) cannot hold.

(2) Subcase $n = 12$. Then $A_{12} \leqslant \Omega_{10}^{+}(3)$. Let $z = \varepsilon_1 + \cdots + \varepsilon_5 - \varepsilon_6 - \cdots - \varepsilon_{10} \in V$. We have $(z,z) = (v,v)$ and $|\langle z\rangle M| = \frac{1}{2}|zM| = \dfrac{12!}{2.(5!)^2.2!} = 8316$. Also $|\langle v\rangle M| = \frac{1}{8}n(n-1)(n-2)(n-3) = 1485$. As $|\mathfrak{E}_1^{+}(V)| = \frac{1}{2}3^{m-1}(3^m-1) = \frac{1}{2}.3^4(3^5-1) = 9801$. Observe $9801 = 8316 + 1485$, hence $\mathfrak{E}_1^{+}(V) = \langle z\rangle M \cup \langle v\rangle M$. Thus M has only two orbits on $\mathfrak{E}_1^{+}(V)$.

(3) Subcase $n = 18$. Then $m = 8, 3^{m-1}+1 = 2188$, and $D = 18 - \varepsilon_3(18) = 17 \equiv -1 = \boxtimes$, so that $\varepsilon = -$, hence $A_{18} \leqslant \Omega_{16}^{-}(3)$. In this case, M has 4 orbits with representatives $v_1 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4, v_2 = \varepsilon_1 + \cdots + \varepsilon_5 - \varepsilon_6 - \varepsilon_7, v_3 = \varepsilon_1 + \cdots + \varepsilon_8 - \varepsilon_9 - \cdots - \varepsilon_{16}$ and $v_4 = \varepsilon_1 + \cdots + \varepsilon_5 - \varepsilon_6 - \cdots - \varepsilon_{10}$. Using GAP, the parameters $c_i, d_i, i = 1, \cdots, 4$ of $\langle v_i\rangle M$ are as follow: $(d_1,c_1) = (3789, 5390), (d_2,c_2) = (223497, 444806), (d_3,c_3) = (328914, 655640)$ and $(d_4,c_4) = (1838565, 3674942)$. We see that $2d_i - c_i = 2188 = 3^{m-1}+1$, for all i, so that equation (3.8) holds with $\varepsilon = -, r = t$ and $m = 8$.

(4) Subcase $n = 24$. Then $c = 15740, d = 16137, c - 2d = -16534, m = 11$ and $D = \boxtimes$, so that $\varepsilon = +$. Then equation (3.1) cannot hold.

(5) Subcase $n = 11$. Then $A_{11} \leqslant \Omega_{10}^{+}(3)$. However as the partition $\lambda = (11,1)$ is a JS-partition and $m(\lambda) \neq \lambda$, by Theorem 2.33, $D^{(11,1)} \downarrow_{A_{11}} \cong D^{(10,1)}$ is irreducible, so that $A_{11} \leqslant A_{12} \leqslant \Omega_{10}^{+}(3)$. Notice that by using GAP, M has 4 orbits on $\mathfrak{E}_1^{+}(V)$ with representatives $v_1 = \varepsilon_1 + \cdots \varepsilon_8 - \varepsilon_9 - \varepsilon_{10}, v_2 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4, v_3 = \varepsilon_1 + \cdots + \varepsilon_5 - \varepsilon_6 - \cdots - \varepsilon_{10}$ and $v_4 = \varepsilon_1 + \cdots + \varepsilon_5 - \varepsilon_6 - \varepsilon_7$ and equation (3.1) holds with $r = s$.

(6) Subcase $n = 17$. Then $m = 8, D = \boxtimes$, and so $\varepsilon = (-)^{m-1} = -$, hence $A_{17} \leqslant \Omega_{16}^{-}(3)$. We have $d = 2823, c = 4816$ and $2d - c = 1330 \neq 3^{m-1}+1$ and $4c$ is not divisible by

$3^{m-1}+1=3^7+12188$. Thus equation (3.9) and (3.8) cannot hold.

(7) Subcase $n=23$. Then $m=11, D=\boxtimes$, and so $\varepsilon=(-)^{m-1}=+$, hence $A_{23}\leqslant\Omega_{22}^{+}(3)$. In this case $3^{m-1}=3^{10}>\frac{1}{8}23.22.21.20=|\langle v\rangle M|$, by (3.10), equation (3.1) cannot hold. ■

Proposition 3.58 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^{\varepsilon}(3)$ and M is almost simple of type A_n with $n\geqslant 12$, and V is not the fully deleted permutation module for A_n in characteristic 3. There is an M-orbit on $\mathfrak{E}_{\xi}^{\varepsilon}(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. By Lemma 2.32, we have $dim(V)\geqslant\frac{1}{2}(n^2-5n+2)$. Hence $2m\geqslant\frac{1}{2}(n^2-5n+2)$, or $m\geqslant\frac{1}{4}(n^2-5n+2)$. However when $n\geqslant 12$, $3^{m-2}\geqslant 3^{\frac{n^2-5n-6}{4}}>n!=|Aut(A_n)|$, which contradicts (3.10) and (3.11). Thus equation (3.1) cannot hold. ■

Proposition 3.59 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^{\varepsilon}(3)$ and M is almost simple of type A_n with $5\leqslant n\leqslant 11$, and V is not the fully deleted permutation module for A_n in characteristic 3. There is an M-orbit on $\mathfrak{E}_{\xi}^{\varepsilon}(V)$ such that equation (3.1) does not hold so that M is not in the Tables 1.1-1.3.*

Proof. By (3.10) and (3.11), we have $dimV\leqslant 2\log_3(n!)+4$. As $m\geqslant 2$, $dimV\geqslant 4$, and so by [26], we only need to consider the following case $(A_n, dimV)=(A_{11}, 34)$. It follows from the Appendix in [24] that the symmetric group S_{11} has exactly two inequivalent irreducible representations of degree 34 in characteristic 3 with corresponding partitions $\lambda=(9,2)$ and $m(\lambda)=(5,4,1^2)$. By Theorem 2.31, $D^{\lambda}\downarrow A_{11}=D^{m(\lambda)}\downarrow A_{11}$ is irreducible. Thus we can assume that $V=D^{\lambda}$. By Theorem 2.25, $dimS^{\lambda}=44$ and by Lemma 2.27, $S^{\lambda}=D^{\lambda}+D^{(10,1)}, dimD^{\lambda}=34$. By Theorem 2.24, S^{λ} has a basis consisting of standard polytabloids. If t is a standard λ-tableaux, then t is one of

$$\begin{matrix} 1 & 2 & j_1 & \cdots & j_7 \\ j_8 & j_9 & & & \end{matrix} \quad \text{or} \quad \begin{matrix} 1 & 3 & j_1 & \cdots & j_7 \\ 2 & j_9 & & & \end{matrix},$$

where $3 \leqslant j_1 < j_2 < \cdots < j_7 \leqslant 11, 3 \leqslant j_8 < j_9 \leqslant 11$, and j_k are all pairwise distinct. Let

$$t_1 = \begin{array}{ccccccccc} 1 & 2 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 4 & & & & & & & \end{array}$$

$$t_2 = \begin{array}{ccccccccc} 1 & 2 & 4 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 5 & & & & & & & \end{array} \text{ and } t_3 = \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 & 11 \\ 5 & 6 & & & & & & & \end{array}.$$

Set $e_i = e_{t_i}$. To simplify the notation, we write just the second row for a tabloid. With this notation, we have $e_1 = \overline{34} - \overline{14} - \overline{32} + \overline{12}$, $e_2 = \overline{35} - \overline{15} - \overline{32} + \overline{12}$, and $e_3 = \overline{56} - \overline{16} - \overline{52} + \overline{12}$. Let $x_1 = e_1 + S^\lambda \cap S^{\lambda\perp}$ and $x_2 = e_1 + e_2 + e_3 + S^\lambda \cap S^{\lambda\perp}$. Then x_i are non-singular in V and $S_{\{7,8,9,10,11\}} \leqslant M_{x_i}$. Thus $|\langle x_i \rangle M| \leqslant |S_{11} : S_{\{7,8,9,10,11\}}| = \frac{11!}{5!}$. As $2m = 34$, $m = 17$. Clearly, $3^{m-2} = 3^{15} > \frac{11!}{5!} \geqslant |\langle x_i \rangle M|$, hence equation (3.1) cannot hold. ■

Cross-characteristic embedding of finite groups of Lie type

Proposition 3.60 *Assume G is nearly simple primitive rank 3 of type $\Omega^\varepsilon_{2m}(3)$ and M is almost simple of type S, where S is a finite simple Chevalley group in cross-characteristic. There is an M-orbit on $\mathfrak{E}^\varepsilon_\xi(V)$ such that equation (3.1) does not hold unless $(S, L) = (L_3(4), \Omega^-_6(3)), (O^+_8(2), O^+_8(3))$, in which cases M has at most two orbits on $\mathfrak{E}^\varepsilon(V)$ so that $1^G_P \not\leqslant 1^G_M$ by Corollary 3.7, and so M is in Table 1.2 or $(L, \widehat{S}) = (\Omega^+_{52}(3), 2.F_4(2))$ is in Table 1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}^\varepsilon_\xi(V)$. Recall the hypothesis in section 3.3, V is an irreducible $\mathbf{F}_3\widehat{S}$-module of even dimension with Frobenius-Schur indicator + and realizes over $\mathbf{F}_3$. Moreover $(3, q) = 1$.

Case $SL_2(q)$. Assume first that $q \equiv 1 (\mathrm{mod}\ 4)$. It follows from Table 2.7(b) that either $dimV = \frac{1}{2}(q+1)$ or $dimV \geqslant q - 1$. However the first case cannot happen as $\frac{1}{2}(q+1) \equiv 1 (\mathrm{mod}\ 2)$, hence $dimV$ is odd. Thus $2m = dimV \geqslant q - 1$. If $q > 29$ then $3^{(q-5)/2} > q^4$ so that $3^{m-2} \geqslant 3^{(q-5)/2} > q^4 \geqslant |Aut(S)|$. By (3.10) and (3.11), equation

(3.1) cannot hold. Thus $5 \leqslant q \leqslant 29, (q, 3) = 1, q \equiv 1 (\text{mod } 4)$ and q is a prime power, hence $q \in \{5, 13, 17, 25, 29\}$. If $q = 29$ then $|Aut(S)| = 29(29^2 - 1) < 3^{(29-1)/2}$, and so $3^{m-2} > 1 + c + d$. Similarly if $q = 5^2$ then $|Aut(S)| = 2.25(25^2 - 1) < 3^{(25-1)/2}$, so that $3^{m-2} > 1 + c + d$. Therefore $q = 5, 13$ or 17. Since $L_2(5) \cong A_5$, we can assume that $q = 13$ or $q = 17$. By [26], $L_2(13)$ has three irreducible representations in characteristic 3 with Frobenius -Schur indicator + and their degrees are 12. However they cannot realize over $\mathbf{F}_3$. Finally by [26] again, we need to consider two cases $(S, dimV) = (L_2(17), 16)$ or $(L_2(17), 18)$. If $(S, dimV) = (L_2(17), 16)$ then $L_2(17) \leqslant P\Omega_{16}^{-}(3)$. For each extension of S, there exist two non-singular points of different type with (c, d) as follow: if $M = L_2(17).2$ then $(c, d) = (32, 120), (722, 501)$; if $M = L_2(17)$ then $(c, d) = (32, 120), (1643, 804)$. If $(S, dimV) = (L_2(17), 18)$ then $L_2(17) \leqslant P\Omega_{18}^{+}(3)$. For each extension of S, there exist two non-singular points of different type with (c, d) as follow: if $M = L_2(17)$ then $(c, d) = (1635, 812), (1611, 836)$; if $M = L_2(17).2$ then $(c, d) = (3185, 1711), (3280, 1615)$. We can check that equation (3.1) cannot hold in any of these cases.

Assume that $q \equiv 3 (\text{mod } 4)$. As q is odd, $dimV = q - 1$ or $dimV = q + 1$ by Table 2.7(c). As above, we have $q \leqslant 29$, and so $q \in \{7, 11, 19, 23\}$. If $q = 23$ then $|Aut(S)| = 23(23^2 - 1) < 3^{(23-1)/2}$, and so $3^{m-2} > 1 + c + d$. Therefore $q = 7, 11$ or 19. By [26], we need to consider the following cases: $(L_2(7), 6), (L_2(11), 10)$. By [9], $\Omega_6^{\varepsilon}(3)$ has no maximal subgroups with socle isomorphic to $L_2(7) \cong L_3(2)$ or its covers. If $(S, dimV) = (L_2(11), 10)$ then $L_2(11) \leqslant P\Omega_{10}^{+}(3)$. For each extension of S, there exist two non-singular points of different type with (c, d) as follow: if $M = L_2(17)$ then $(c, d) = (20, 45), (116, 48)$; if $M = L_2(17).2$ then $(c, d) = (3185, 1711), (443, 216)$. We can check that equation (3.1) cannot hold in any of these cases.

Assume that $q \equiv 0 (\text{mod } 2)$. As q is even, $q \pm 1$ are odd, by Table 2.7(d), $dimV = q$ and $3 \nmid q + 1$. As $L_2(4) \cong A_5$, we need to consider case $(L_2(16), 16)$. If $(S, dimV) = (L_2(16), 16)$ then $L_2(16) \leqslant P\Omega_{16}^{-}(3)$ and $Out(L_2(16)) \cong \mathbb{Z}_4$. For each extension of S, there exist two non-singular points of different type with (c,d) as follow: if $M = L_2(16)$ then $(c,d) = (1355, 648), (30, 105)$; if $M = L_2(16).2$ then $(c,d) = (30, 105), (620, 399)$; if $M = L_2(16).4$ then $(c,d) = (30, 105), (5378, 2781)$. We can check that equation (3.1) cannot hold in any of these cases.

Case $L_n(q), n \geqslant 3$. Assume $(n,q) \neq (3,2), (3,4), (4,2)$. By Table 2.6, $2m \geqslant (q^n - 2q + 1)/(q-1)$. Now, if $(n,q) \neq (3,2), (4,2), (5,2), (3,4), (3,5)$, then $3^{m-2} > q^{n^2} \geqslant |Aut(L_n(q))|$ so that (3.1) cannot hold. By the isomorphisms $L_2(7) \cong L_3(2)$, and $L_4(2) \cong A_8$ we only need to consider the following cases: $L_5(2), L_3(4), L_3(5)$. If $S = L_3(4)$, then by [26], either $dimV = 6$ or $dimV \geqslant 36$. If the latter case holds then $3^{m-2} \geqslant 3^{16} > 4^9 \geqslant |Aut(L_3(4))|$, hence (3.1) cannot hold. By [9], $L_3(4)$ embeds in $\Omega_6^-(3)$ and $L_3(4)$ has only one orbit on $\mathfrak{E}^\varepsilon(V)$. If $S = L_3(5)$, then by [26], $dimV = 2m \geqslant 30$. As $3^{m-2} \geqslant 3^{13} > 2.5^8 \geqslant |Aut(L_3(5))| = 2.5^3(5^2-1)(5^3-1)$, hence (3.1) cannot hold. If $S = L_5(2)$, then by [26], either $dimV = 30$ or $dimV \geqslant 124$. If the latter case holds then $3^{m-2} \geqslant 3^{60} > 2^{25} = |Aut(L_5(2))|$, hence (3.1) cannot hold.

Case $PSp_{2n}(q)$, q odd. By Table 2.6, $2m \geqslant \frac{1}{2}(q^n - 1)$. If $(n,q) \neq (2,5), (2,7), (3,5)$, then $3^{m-2} \geqslant 3^{(q^n-9)/4} > q^{2n^2+n+1} \geqslant |Aut(S)|$, so that (3.1) cannot hold. Thus, we need to consider cases $S_4(5), S_4(7), S_6(5)$. If $S = S_6(5)$, then by Theorem 2.1 in [16], either $dimV = \frac{1}{2}(q^n \pm 1) = 62, 63$ or $dimV \geqslant (q^n-1)(q^n-q)/(2q+2) = 1240$. By [19], if $dimV = 62$, then $ind(V) = -$. Thus $dimV \geqslant 1240$, hence $3^{m-2} > |Aut(S)|$. If $S = S_4(7)$ then $dimV \geqslant 126$ by [19] and so $3^{m-2} > |Aut(S)|$. Finally if $S = S_4(5)$ then $dimV \geqslant 40$ by [26]. Then $3^{m-2} \geqslant 3^{18} > 5^{11} \geqslant |Aut(S)|$, so that (3.1) cannot happen.

Case $PSp_{2n}(q)$, q even. As $Sp_4(2)' \cong A_6$, we can assume that $(n,q) \neq (2,2)$. By Table 2.6, $2m \geqslant (q^n-1)(q^n-q)/(2q+2)$. If $(n,q) \neq (2,2), (3,2), (4,2), (2,4)$ then $3^{m-2} > q^{2n^2+n+1} \geqslant |Aut(S)|$, so that equation (3.1) cannot hold. Thus, we need to consider cases: $S_4(4), S_6(2), S_8(2)$. By [26] and [19], we need to consider the following cases: $(S_4(4), 18), (S_4(4), 34), (S_4(4), 50), (S_6(2), 14), (S_6(2), 34), (S_8(2), 50)$. If S is one of $S_8(2)$

or $S_4(4)$ and $dimV = 50$, then $m = 25$, $|Aut(S_8(2))| = 2^{16}.(2^2-1)(2^4-1)(2^6-1)(2^8-1) < 2^{16+20} = 2^{36}$, $|Aut(S_4(4))| = 2.4^4(4^2-1)(4^4-1) < 2.4^{10} = 2^{21}$. Now, as $3^{23} > 2^{36} > 2^{21}$, we have $3^{m-2} = 3^{23} > |Aut(S_8(2))| > |Aut(S_4(4))|$. Similarly if $S = S_6(2)$ or $S_4(4)$ and $dimV = 34$, then $m = 17$, $|Aut(S_6(2))| = 2^9(2^2-1)(2^4-1)(2^6-1) < 2^{21}$. We have $3^{15} > 2^{21}$ so that $3^{m-2} > |Aut(S_6(2))|$ and $3^{m-2} > |Aut(S_4(4))|$. Thus (3.1) cannot hold in any of these cases.

Case $U_n(q)$, $(q,3) = 1, n \geqslant 3, n$ odd. By Table 2.6, $2m \geqslant (q^n - q)/(q+1)$. If $(n,q) \neq (3,2),(5,2),(7,2),(3,4),(3,5)$ then $3^{m-2} > |Aut(S)|$, so that equation (3.1) cannot hold. Since $U_3(2) \cong 3^2.Q_8$ is not simple, we need to consider cases: $U_5(2), U_7(2), U_3(4), U_3(5)$. If $S = U_3(4)$, then by [26], $dimV = 64$. But then $3^{m-2} > |Aut(S)|$. If $S = U_3(5)$, then by [26], either $dimV = 28$ or $dimV \geqslant 84$. If the latter case holds then $3^{m-2} \geqslant 3^{40} > 5^9 \geqslant |Aut(U_3(5))|$, hence equation (3.1) cannot hold. If $S = U_5(2)$ then by [26], $dimV = 44$. As $3^{m-2} = 3^{20} > 2^{25} \geqslant |Aut(S)|$. If $S = U_7(2)$, then by [19], $dimV > 250$. Then $3^{m-2} > |Aut(S)|$.

Case $U_n(q)$, $n \geqslant 3$, even. Assume $(n,q) \neq (4,2)$. By Table 2.6, $2m \geqslant (q^n-1)/(q+1)$. If $(n,q) \neq (6,2)$ then $3^{m-2} > |Aut(S)|$, so that equation (3.1) cannot hold. As $U_4(2) \cong S_4(3) \cong O_5(3)$, we need to consider cases $U_6(2)$. By [26], $dimV \geqslant 56$. But then $3^{m-2} = 3^{26} > 2^{36} \geqslant |Aut(S)|$.

Case $P\Omega_{2n}^+(q), n \geqslant 4$. Assume $q > 3$. By Table 2.6, $2m \geqslant (q^n-1)(q^{n-1}+q)/(q^2-1)-2$. As $|Aut(S)| \leqslant 3q^{2n^2-n+1} < q^{2n^2-n+2}$, $m-2 \geqslant (q^n-1)(q^{n-1}+q)/(2q^2-2)-3$ and $(q^n-1)(q^{n-1}+q)/(2q^2-2)-3 > (2n^2-n+2)\log_3(q)$ for any $n \geqslant 4, q \geqslant 4$. It follows that $3^{m-2} > |Aut(S)|$. Assume that $q = 2$ and $n \geqslant 5$. By Table 2.6 again, $dimV = 2m \geqslant (q^n-1)(q^{n-1}-1)/(q^2-1)$. Then $|Aut(S)| \leqslant 2^{2n^2-n+1}$ and $m-2 \geqslant (q^n-1)(q^{n-1}-1)/(2q^2-2)-2$. We have $(q^n-1)(q^{n-1}-1)/(2q^2-2)-2 > (2n^2-n+1)\log_3(2)$, so that $3^{m-2} > |Aut(S)|$. Thus we are left with $S = P\Omega_8^+(2)$. By [9], either $dimV = 8, 28$ or $dimV \geqslant 48$. Suppose that the latter case holds. As $|Aut(S)| \leqslant 3.2^{29}$ and $m \geqslant 24$, we have $3^{m-2} \geqslant 3^{22} > 3.2^{29} \geqslant |Aut(P\Omega_8^+(2))|$.

If $(S, dimV) = (P\Omega_8^+(2), 28)$ then $P\Omega_8^+(2) \leqslant P\Omega_{28}^+(3)$. In this case, there exist two non-singular points $\langle u_i \rangle, i = 1, 2$, of different type with orbit sizes $3780, 45360$ respectively. Clearly $|\langle u_i \rangle M| < 3^{12} = 3^{m-2}$. By (3.10) and (3.11), equation (3.1) cannot hold.

If $dimV = 8$ then by [9], $S = O_8^+(2)$ is a maximal subgroup of $L = O_8^+(3)$, and there are 3 classes of S in L. By [13] again, the permutation characters $1_S^L = 1a + 260cde + 9450a + 18200e, 1a + 260bdf + 9450a + 18200d, 1a + 260aef + 9450a + 18200c$, respectively, while $1_M^L = 1a + 260a + 819a, 1a + 260d + 819a, 1a + 260b + 819b, 1a + 260e + 819b, 1a + 260c + 819c, 1a + 260f + 819c$. Thus M has at most two orbits on $\mathfrak{E}^\varepsilon(V)$.

Case $P\Omega_{2n}^-(q), n \geqslant 4$. Assume that $(n, q) \neq (4, 2), (4, 4), (5, 2)$. By Table 2.6, $2m \geqslant (q^n + 1)(q^{n-1} - q)/(q^2 - 1) - 1$. As $|Aut(S)| \leqslant (q^n + 1)q^{2n^2-n+1} < q^{2n^2-n+2}$, $m - 2 \geqslant (q^n+1)(q^{n-1}-q)/(2q^2-2)-3$ and $(q^n+1)(q^{n-1}-q)/(2q^2-2)-3 > (2n^2-n+2)\log_3(q)$ for any (n, q) as above. It follows that $3^{m-2} > |Aut(S)|$ for (n, q) as above. If $S = P\Omega_8^-(4)$ then $dimV \geqslant 1026$, hence $3^{m-2} \geqslant 3^{511} > 4^{30} > |Aut(S)|$. If $S = P\Omega_{10}^-(2)$ then $dimV \geqslant 186$ by [19], and hence $3^{m-2} > |Aut(S)|$. Finally if $S = P\Omega_8^-(2)$ then either $dimV = 34$ or $dimV \geqslant 50$ by [26]. Clearly if $dimV \geqslant 50$ then $3^{m-2} > |Aut(S)|$. If $(S, dimV) = (P\Omega_8^-(2), 34)$ then $P\Omega_8^-(2) \leqslant P\Omega_{34}^-(3)$. In this case, there exist two non-singular points $\langle u_i \rangle, i = 1, 2$, of different type which are the eigenvector of an element g of order 17. The normalizer N in S of the subgroup generated by g is of order 68, and N fixes $\langle u_i \rangle, i = 1, 2$. We have $|\langle u_i \rangle M| \leqslant |Aut(S)|/|N| < 3^{15} = 3^{m-2}$, in view of (3.10) and (3.11), equation (3.1) cannot hold.

Case $P\Omega_{2n+1}(q), n \geqslant 3, q > 3, q$ odd. By Table 2.6, $2m \geqslant (q^{2n} - 1)/(q^2 - 1) - 2$. As $|Aut(S)| \leqslant q^{2n^2+n+1}$, $m - 2 \geqslant (q^{2n} - 1)/(2q^2 - 2) - 3$ and $(q^{2n} - 1)/(2q^2 - 2) - 3 > (2n^2 + n + 2)\log_3(q)$ for any (n, q) as above. It follows that $3^{m-2} > |Aut(S)|$.

Case $E_6(q)$. By Table 2.6, $2m \geqslant q(q^4+1)(q^6+q^3+1)-2$. As $|Aut(S)| \leqslant q^{79}$, $m-2 \geqslant q(q^4 + 1)(q^6 + q^3 + 1)/2 - 3 > 79.\log_3(q)$ for any $q \geqslant 2$. It follows that $3^{m-2} > |Aut(S)|$.

Case $E_7(q)$. By Table 2.6, $2m \geqslant e(S)$, where $e(S) = q\Phi_7(q)\Phi_{12}(q)\Phi_{14}(q) - 3 = q(q^6 + \cdots + q + 1)(q^6 - q^5 + q^4 - q^3 + q^2 - q + 1) - 3$. As $|Aut(S)| \leqslant q^{134}$, $m - 2 \geqslant \frac{1}{2}(e(S) - 4) > 134.\log_3(q)$ for any $q \geqslant 2$. It follows that $3^{m-2} > |Aut(S)|$.

Case $E_8(q)$. By Table 2.6, $2m \geqslant e(S)$, where $e(S) \geqslant q^{27}(q^2 - 1)$. As $|Aut(S)| \leqslant q^{249}$, $m - 2 \geqslant \dfrac{e(S) - 4}{2} > 249.\log_3(q)$ for any $q \geqslant 2$. It follows that $3^{m-2} > |Aut(S)|$.

Case $F_4(q), q$ odd. As q is odd and $(q, 3) = 1$, it follows that $q \geqslant 5$. We have $2m \geqslant q^4(q^6 - 1)$ and $|Aut(S)| \leqslant q^{53}$. Since $m - 2 \geqslant \frac{1}{2}q^4(q^6 - 1) - 2 > 53.\log_3(q)$ for any $q \geqslant 5$, hence $3^{m-2} > |Aut(S)|$.

Case $F_4(q), q$ even. Assume that $q > 2$. Then $q \geqslant 4$. Thus by Table 2.6, $2m \geqslant e(S) = \frac{1}{2}q^7(q^3 - 1)(q - 1)$. Now, as $m - 2 \geqslant \frac{1}{2}e(S) - 2 > 53\log_3(q)$ for any $q > 2$, and so $3^{m-2} > |Aut(S)|$ if $q \geqslant 4$. If $q = 2$ then by [19], either $dimV = 52$ or $dimV > 250$. If the latter case holds then clearly, $3^{m-2} > |Aut(S)|$. Thus we only need to consider case $dimV = 52$ and $\widehat{S} \cong 2.F_4(2)$.

Case $G_2(q)$. As $G_2(2) \cong U_3(3) \cdot 2$ and $(q, 3) = 1$, we can assume that $q \geqslant 4$. Suppose first that $q > 4$. Then by Table 2.6, $2m \geqslant q(q^2 - 1)$. Since $|Aut(G_2(q))| \leqslant q^{15}$, and $m - 2 \geqslant \frac{1}{2}q(q^2 - 1) - 2 > 15\log_3(q)$ for any $q > 4$, we have $3^{m-2} > |Aut(S)|$ when $q \geqslant 4$. If $S = G_2(4)$ then by [26], $dimV \geqslant 64$ and hence $3^{m-2} \geqslant 3^{30} > 4^{15} \geqslant |Aut(S)|$.

Case ${}^2E_6(q)$. By Table 2.6, $2m \geqslant q^9(q^2 - 1)$. Since $|Aut(S)| \leqslant q^{79}$, and $m - 2 \geqslant \frac{1}{2}q^9(q^2 - 1) - 2 > 79\log_3(q)$, $3^{m-2} > |Aut(S)|$.

Case ${}^3D_4(q)$. By Table 2.6, $2m \geqslant q^3(q^2 - 1)$. Since $|Aut(S)| \leqslant 3.q^{29}$, and $m - 2 > \frac{1}{2}q^3(q^2 - 1) - 3 > 29\log_3(q)$ for any $q \geqslant 3$, $3^{m-2} > |Aut(S)|$ whenever $q > 2$. If $S = {}^3D_4(2)$ then $dimV \geqslant 52$ by [26], and hence $3^{m-2} \geqslant 3^24 > 3.2^29 > |Aut(S)|$.

Case ${}^2F_4(q), q = 2^{2a+1}$. By Table 2.6, $2m \geqslant q^4.\sqrt{q/2}(q - 1)$. Since $|Aut(S)| \leqslant q^{27}$, and $m - 2 \geqslant \frac{1}{2}q^4.\sqrt{q/2}(q - 1) - 2 > 27\log_3(q)$ for any $q \geqslant 8$, $3^{m-2} > |Aut(S)|$. We are left with case $S = {}^2F_4(2)'$. By [19], $dimV \geqslant 124$, so that $3^{m-2} > |Aut(S)|$.

Case $Sz(q), q = 2^{2a+1}$. As $Sz(2)$ is not simple, we can assume that $q \geqslant 8$. Suppose first that $q > 8$. By Table 2.6, $2m \geqslant \sqrt{q/2}(q-1)$. Since $|Aut(S)| \leqslant q^6$, and $m-2 \geqslant \frac{1}{2}\sqrt{q/2}(q-1)-2 > 6\log_3(q)$, $3^{m-2} > |Aut(S)|$. If $S = Sz(8)$ then $dimV \geqslant 40$ by [19]. Then $3^{m-2} \geqslant 3^{18} > 8^6 = 2^{18} \geqslant |Aut(S)|$. ■

Defining characteristic embedding of finite groups of Lie type

Proposition 3.61 *Assume G is nearly simple primitive rank 3 of type $\Omega^{\varepsilon}_{2m}(3)$ and M is almost simple of type S, where $\widehat{S}$ is simply connected of type A_ℓ or ${}^2A_\ell$ over $\mathbf{F}_{3^f}$. There is an M-orbit on $\mathfrak{E}^{\varepsilon}_{\xi}(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3 .*

Proof. Assume (3.1) holds for some $r \in \{s,t\}$ and for any points in $\mathfrak{E}^{\varepsilon}(V)$. Let $\beta = \{e_1, \cdots, e_{\ell+1}\}$ be a basis for N, the natural module of $\widehat{S}$ over $\mathbf{F}_{3^f}$.

Case $f = 1$. We can assume that $\ell \geqslant 2$. By Theorem 2.35 and 2.36, there exists a 3-restricted dominant weight $\lambda \in X_3$ such that $V \cong L(\lambda)$. By Corollary 3.36, and the fact that $1+c+d \leqslant |Aut(S)| \leqslant 3^{(\ell+1)^2}$, we have $3^{m-2} \leqslant 3^{(\ell+1)^2}$, and hence $m-2 \leqslant (\ell+1)^2$, so that $2m \leqslant 2\ell^2+4\ell+6$. If $\ell \geqslant 18$, then $\frac{1}{8}\ell^3 \geqslant 2\ell^2+4\ell+6$, and so by Theorem 5.1 in [38], λ is one of the following 3-restricted dominant weights $\{\lambda_1, \lambda_\ell, \lambda_2, \lambda_{\ell-1}, 2\lambda_1, 2\lambda_\ell, \lambda_1+\lambda_\ell\}$. Since $L(\lambda)$ is self-dual and $\ell \geqslant 18$, by Proposition 2.38, $\lambda = \lambda_1+\lambda_\ell$. If $\ell < 18$, then by Theorem 4.4, Appendix A_6 through A_{21} in [38], either $\lambda = \lambda_1+\lambda_\ell$ or $\lambda = \lambda_{(\ell+1)/2}$, for $\ell = 3, 5, 7$.

As $dimL(\lambda_1+\lambda_\ell) = \ell^2+2\ell-\varepsilon_3(\ell+1)$, it follows that $dimL(\lambda_1+\lambda_\ell) = 2m$ is even if and only if $\ell = 6k-2, 6k-1, 6k$ for some positive integer k. Using the constructions prior to Proposition 3.28, $L(\lambda_1+\lambda_\ell) \cong V := V_1/(V_1 \cap V_2)$. Let U be the subgroup of $\widehat{S}$ consisting of all matrices of the form $diag(I_2, A)$, where $A \in SL^{\varepsilon}_{\ell-1}(3)$. Then $U \cong SL^{\varepsilon}_{\ell-1}(3)$. For $\xi = \pm 1$, let $x_\xi = E_{1,2} + \xi E_{2,1} + V_1 \cap V_2$, when $\varepsilon = +$, and $x_+ = E_{1,2} + E_{2,1} + V_1 \cap V_2, x_- = \mu E_{1,2} + \overline{\mu} E_{2,1} + V_1 \cap V_2$ when $\varepsilon = -$. Then $x_\xi \in V$ and $Q(x_\xi) \neq 0$. It follows that $\langle x_\xi \rangle$ is a non-singular point in V, of plus or minus type depending on ξ and ℓ. As $V_1 \cap V_2$ is fixed under natural action of U, and clearly, U centralizes x_ξ, it follows that $U \leqslant \widehat{S}_{\langle x_\xi \rangle}$, the stabilizer of $\langle x_\xi \rangle$ in $\widehat{S}$. We have $3^{m-2} \leqslant |M : M_{\langle x_\xi \rangle}| \leqslant |Aut(S) : U| = [Aut(L^\varepsilon_{\ell+1}(3)) : SL^\varepsilon_{\ell-1}(3)] < 3^{4\ell+2}$. We deduce that $2m < 8\ell + 8$. If $\ell \geqslant 8$, then $\ell^2 + 2\ell - 1 > 8\ell + 8$, so that $2m = \ell^2 + 2\ell - \varepsilon_3(\ell + 1) \geqslant \ell^2 + 2\ell - 1 > 8\ell + 8$, which is a contradiction. Thus we assume that $2 \leqslant \ell \leqslant 7$. However since ℓ has the form $6k - 2, 6k - 1, 6k$, ℓ must be one of the following values $\{4, 5, 6\}$. Computation shows that with $n = \ell + 1$:

$$1 + c_{x_\xi} + d_{x_\xi} = \frac{3^{2n-3}(3^n - \varepsilon^n)(3^{n-1} - \varepsilon^{n-1})}{(6 - 2\varepsilon)(3 - \xi)}$$

$$\begin{aligned} d_\xi &= \frac{3^{2n-4}(3-\xi)}{2} - \frac{(1+\xi\varepsilon)(3^{n-2} - \varepsilon^{n-2})3^{2n-4}}{3-\varepsilon} \\ &+ \frac{3^{2n-4}((3^n - \varepsilon^n)(3^{n-1} - \varepsilon^{n-1}) + 2(3^{n-2} - \varepsilon^{n-2})(3^{n-3} - \varepsilon^{n-3}) - (3^2 - \varepsilon^2)(3 - \varepsilon))}{(3-\xi)(6-2\varepsilon)} \\ c_\xi &= \frac{3^{2n-3}(3^n - \varepsilon^n)(3^{n-1} - \varepsilon^{n-1})}{(6-2\varepsilon)(3-\xi)} - \frac{3^{2n-4}(3-\xi)}{2} + \frac{3^{2n-4}(1+\xi\varepsilon)(3^{n-2} - \varepsilon^{n-2})}{3-\varepsilon} - 1 \\ &- \frac{3^{2n-4}((3^n - \varepsilon^n)(3^{n-1} - \varepsilon^{n-1}) + 2(3^{n-2} - \varepsilon^{n-2})(3^{n-3} - \varepsilon^{n-3}) - (9 - \varepsilon^2)(3 - \varepsilon)}{(6-2\varepsilon)(3-\xi)} \end{aligned}$$

We can check that equation (3.1) cannot hold in any of these cases.

If $\ell = 3, \lambda = \lambda_2$ then $dimL(\lambda_2) = 6$, and $L_4(3) \cong P\Omega_6^+(3)$. If $\ell = 5, \lambda = \lambda_3$, then $V \cong \bigwedge^3 N$ and $dimV = 20$. However, in this case, the Frobenius-Schur indicator of V is -1 so that $\widehat{S}$ does not leave invariant a non-degenerate symmetric form on V. Finally if $\ell = 7, \lambda = \lambda_4$, then $V \cong \bigwedge^4 N$ and $dimV = 70$. Then V has a basis consisting of $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$, where $1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant \ell + 1$. Fix an isomorphism $\bigwedge^{\ell+1} N$ to $\mathbf{F}_3$. The bilinear form on V is the projection of $w_1 \wedge w_2$ to $\bigwedge^{\ell+1} N$ for any $w_1, w_2 \in V$. Let $w = e_1 \wedge e_2 \wedge e_3 \wedge e_4 + \xi e_5 \wedge e_6 \wedge e_7 \wedge e_8, \xi = \pm 1$. Then w is a non-singular vector in V and $(SL(\langle e_1, \cdots, e_4 \rangle) \times SL(\langle e_5, \cdots, e_8 \rangle)).2 \cong (SL_4(3) \times SL_4(3)).2$ fixes $\langle w \rangle$. Thus $|Aut(L_8(3)) : (SL_4(3) \times SL_4(3)).2| \leqslant 3^{16}(3^5 + 1)(3^6 - 1)(3^7 + 1)(3^8 - 1)/(8.26.80) < (3^{34} + 1)/2$. Hence $1 + c + d < (3^{m-1} + 1)/2$ so that equation (3.1) cannot hold.

Case $f > 1$. We first consider case $\ell = 1$. Since $SL_2(q) \cong SU_2(q) \cong Sp_2(q)$, we can assume that $\varepsilon = +$. Moreover, as $L_2(9) \cong A_6$, we also assume that $f \geqslant 3$. If λ is any 3-restricted dominant weight then $\lambda = c\lambda_1$, where $1 \leqslant c \leqslant 2$, $dimL(c\lambda_1) = c+1$ and $L(c\lambda_1)$ is self-dual. By Proposition 2.41, $dimV = (dim\Psi)^f$, for some irreducible kS-module Ψ. Clearly $\Psi = L(\lambda)$ for some $\lambda \in X^+$, hence $dimV \geqslant (dim\Psi)^f \geqslant 2^f$. We have $|Aut(L_2(3^f))| \leqslant 3^{4f}$ and $m-2 \geqslant 2^{f-1}-2$. If $f \geqslant 6$, then $2^{f-1}-2 > 4f$ and hence $m-2 > 4f$ so that $3^{m-2} > 3^{4f}$. By (3.10) and (3.11), equation (3.1) cannot hold. Thus, we can assume that $3 \leqslant f \leqslant 5$. As $dimV$ is even, $dim\Psi$ is also even. If $dim\Psi > 2$ then $dim\Psi \geqslant 4$. Then $dimV = 2m \geqslant 4^f$, so that $m-2 \geqslant 2^{2f-1}-2$, and also $m-2 \geqslant 2^{2f-1}-2 > 4f$. Thus equation (3.1) cannot hold. Therefore $dim\Psi = 2$, and hence $\Psi = L(\lambda_1)$. Since Ψ is invariant under a non-degenerate alternating form, it follows that V is invariant under a product of f alternating forms, hence it fixes a non-degenerate symmetric form if and only if f is even. Therefore, we conclude that $f = 4$ and $S = SL_2(2^4)$. Now, we consider case $\ell \geqslant 2$. It follows from the case $f = 1$ that if Ψ is an irreducible $k\widehat{S}$-module which leaves invariant a non-degenerate symmetric form then either $dim\Psi \geqslant \ell^2+2\ell-\varepsilon_3(\ell+1) \geqslant \ell^2+2\ell-1$ or $\ell = 3,5$ and $dim\Psi \geqslant 6, 20$, respectively. By Proposition 2.41, $dimV = (dim\Psi)^f$, for some irreducible $k\widehat{S}$-module Ψ. Hence $dimV \geqslant (\ell^2+2\ell-1)^f$ if $\ell \neq 3,5$ and $dimV \geqslant 6^f, 20^f$ if $\ell = 3,5$, respectively. Using (3.10) and (3.11), we need to consider the case $\widehat{S} = SL_4^{\varepsilon}(9)$ and $dimW = 36$. However as $L_4^{\varepsilon}(9) \cong P\Omega_6^{\varepsilon}(9)$, and $L_2(3^4) \cong \Omega_4^-(9)$. The results follow by Proposition 3.63. ■

We next consider the following embedding of classical groups by using twisted tensor products: $Cl_n(q^\alpha) \hookrightarrow Cl_{n^\alpha}(q)$. These embedding can arise as follows, for example, take $S = PSp_{2\ell}(3^2)$. Let $(N, \mathbf{F}_{3^2}, \mathbf{f}_1)$ be a classical symplectic geometry. For any $a \in \mathbf{F}_{3^2}$, denote by $\overline{a}$ the image under the involutory field automorphism of $\mathbf{F}_{3^2}$. Let $\mathbf{f}_2$ be the non-degenerate symplectic form on $N^{(1)}$. Then $\widehat{S}$ preserves a non-degenerate quadratic form Q on $N \otimes N^{(1)}$ which satisfies $Q(v \otimes w) = 0$ for $v, w \in N$ and $Q(v+w) = Q(v)+Q(w)+\mathbf{f}(v,w)$, where $\mathbf{f}$ is the product form $\mathbf{f}_1\mathbf{f}_2$. Let $\beta_0 = \{v_1, v_2, \cdots, v_{2\ell-1}, v_{2\ell}\}$ be a basis for N such that

$\mathbf{f}_1(v_{2i-1}, v_{2i}) = 1, \mathbf{f}_1(v_{2i-1}, v_{2j-1}) = 0 = \mathbf{f}_1(v_{2i}, v_{2j}), 1 \leqslant i, j \leqslant \ell$. Let μ be a generator for $\mathbf{F}_{3^2}^*$, and let $\beta = \{v_i \otimes v_i, v_j \otimes v_k + v_k \otimes v_j, \mu v_j \otimes v_k + \overline{\mu} v_k \otimes v_j, 1 \leqslant i \leqslant 2\ell, 1 \leqslant j < k \leqslant 2\ell\}$. By Proposition 2.4 and 2.5 in [41], $\widehat{S}$ fixes the non-degenerate symmetric form $\mathbf{f}$ on $V = span_{\mathbf{F}_3}(\beta)$, so that $S \leqslant P\Omega^{\varepsilon}_{(2\ell)^2}(3)$ where $\varepsilon = (-)^{\ell}$. With these results and notations, we have:

Proposition 3.62 *Suppose that $S \cong PSp_{2\ell}(3^2)$. Embed S into $P\Omega^{\varepsilon}_{(2\ell)^2}(3)$ via V, where $\varepsilon = (-)^{\ell}$. Assume that $\ell \geqslant 4$. There is an M-orbit on $\mathfrak{C}^{\varepsilon}_{\xi}(V)$ such that equation (3.1) does not hold.*

Proof. For $\xi \in \{\pm 1\}$, let $z_\xi = v_1 \otimes v_1 + \xi v_2 \otimes v_2 \in V$. As $Q(z_\xi) = Q(v_1 \otimes v_1) + Q(\xi v_2 \otimes v_2) + f(v_1 \otimes v_1, \xi v_2 \otimes v_2) = \xi \neq 0$, z_ξ are non-singular in V. Let $U = \{v_1, v_2\} \leqslant N$ and W_2 be the $\mathbf{F}_3$ span of $\{v_1 \otimes v_1, v_2 \otimes v_2, v_1 \otimes v_2 + v_2 \otimes v_1, \mu v_1 \otimes v_2 + \overline{\mu} v_2 \otimes v_1\}$. Then $PSp(U) \cong PSp_2(9)$ embeds into $P\Omega_4^-(3)$ as argument above, and by comparing the order, we have an isomorphism. It follows that the stabilizer in $PSp(U)$ of the point $\langle z_\xi \rangle$ is isomorphic to $\Omega_3(3)$. Clearly $PSp(U^\perp) \cong PSp_{2\ell-2}(9)$ fixes z_ξ, hence $\Omega_3(3) \times PSp_{2\ell-2}(3) \leqslant S_{\langle z_\xi \rangle}$. Therefore $[Aut(PSp_{2\ell}(9)) : (\Omega_3(3) \times PSp_{2\ell-2}(9))] = 3^{4\ell-3}(3^{4\ell}-1) \leqslant 3^{8\ell-3}$. As $2m = (2\ell)^2$, $m - 2 = 2\ell^2 - 2$. Observe that $m - 2 - (8\ell - 3) = 2\ell^2 - 2 - 8\ell + 3 = 2\ell(\ell - 4) + 1 > 0$ as $\ell \geqslant 4$. Thus $3^{m-2} > 3^{8\ell-3}$, and so equation (3.1) cannot hold by (3.10) and (3.11). ■

Next we exhibit the embedding of $P\Omega^{\pm}_{2\ell}(3^2)$ into $P\Omega^{\varepsilon}_{(2\ell)^2}(3)$. Let λ be a root of $x^2 - x - 1$ in $\overline{\mathbf{F}}_3$. Then λ is a generator for $\mathbf{F}_9^*$. Let $\mu = \lambda^2$. We have $\mu^2 = -1$ and $\mu^3 = -\mu$. Let $(V, \mathbf{F}_9, Q)$ be a classical orthogonal geometry of type ξ. First assume $\xi = +$. As $\frac{1}{2}\ell(9-1) = 2\ell$ is even, by Proposition 2.6(i), $D(Q) = \square$, hence N has a basis β such that f_β is $I_{2\ell}$. By Proposition 2.7 in [41], $N \otimes N^{(1)}$ has a basis β consisting of

$$\begin{aligned} u_i &= v_i \otimes v_i \\ a_{ij} &= v_i \otimes v_j + v_j \otimes v_i, i < j \\ b_{ij} &= \mu v_i \otimes v_j - \mu v_j \otimes v_i, i < j \end{aligned}$$

where $1 \leqslant i, j \leqslant 2\ell$. Moreover $\varepsilon = (-)^{\ell}$ and all the values of the product form on this module lie in $\mathbf{F}_3$, so that $S = \Omega^{+}_{2\ell}(3^2)$ preserves a non-degenerate symmetric form on $W = span_{\mathbf{F}_3}(\beta)$, hence $\overline{S}$ embeds into $P\Omega^{\varepsilon}_{(2\ell)^2}(3)$. Next, assume that $\xi = -$. As $\frac{1}{2}\ell(9-1) = 2\ell$ is even, by Proposition 2.6,(ii), $D(Q)$ is a non-square, and hence by (iv), N has a basis β such that f_β is $diag(\lambda, 1, \cdots, 1)$. By Proposition 2.7 in [41] again, chose a basis β $N \otimes N^{(1)}$ which consists

$$\begin{array}{rclrcl} u_1 & = & \mu\lambda^{-1} v_1 \otimes v_1, & u_i & = & v_i \otimes v_i, i > 1 \\ a_{1j} & = & x v_i \otimes y v_j + v_j \otimes v_i, 1 < j, & a_{ij} & = & v_i \otimes v_j + v_j \otimes v_i, i < j \\ b_{1j} & = & y v_i \otimes v_j - x v_j \otimes v_i, 1 < j, & b_{ij} & = & \mu v_i \otimes v_j - \mu v_j \otimes v_i, i < j \end{array}$$

where $1 \leqslant i, j \leqslant 2\ell$, and $x^2 + y^2 = \lambda^{-1}$, also $\varepsilon = (-)^{\ell}$. As above, we have an embedding $P\Omega^{-}_{2\ell}(3^2) \leqslant P\Omega^{\varepsilon}_{(2\ell)^2}(3)$ via the module $V = span_{\mathbf{F}_3}(\beta)$. We will use the usual notation $(.,.)$ for the product form on $N \otimes N^{(1)}$.

Proposition 3.63 *Let $S = P\Omega^{\xi}_{2\ell}(3^2)$. Embed S into $P\Omega^{\varepsilon}_{(2\ell)^2}(3)$ via V, where $\varepsilon = (-)^{\ell}$. Assume that $\ell \geqslant 3$. There is an M-orbit on $\mathfrak{E}^{\varepsilon}_{\xi}(V)$ such that equation (3.1) does not hold.*

Proof. Case $\xi = +$. Let $x = v_1 \otimes v_1$. Then $(x, x) = (v_1 \otimes v_1, v_1 \otimes v_1) = (v_1, v_1)(v_1, v_1) = 1$. Hence x is non-singular in V. Let H be the stabilizer in S of $\langle v_1 \rangle$. It follows from Proposition 4.1.6 in [29] that $H \cong \langle -1 \rangle \times \Omega_{2(\ell-1)+1}(9)$. Therefore $|\langle v_1 \rangle S| = |S : H| = \frac{1}{2}9^{\ell-1}(9^{\ell} - 1) = \frac{1}{2}3^{2\ell-2}(3^{2\ell} - 1)$. Observe that if g fixes $\langle v_1 \rangle$ then g^{ν} also fixes $\langle v_1 \rangle$. Thus g fixes $x = v_1 \otimes v_1$ if $g \in H$. It follows that $H \leqslant S_{\langle x \rangle}$. However, as H is maximal in S, this forces $H = S_{\langle x \rangle}$. We have $A = |M : M_{\langle x \rangle}| \leqslant |Aut(S) : H| = |Out(S)||S : H| \leqslant 3.8.9^{\ell-1}(9^{\ell} - 1) < 3^{4\ell+1}$ since $|Out(S)| \leqslant 48$. As $2m = (2\ell)^2$, $m - 2 = 2\ell^2 - 2$. Now, since $\ell \geqslant 3$, $m - 2 - 4\ell - 1 = 2\ell^2 - 4\ell - 3 = 2\ell(\ell - 2) - 3 > 0$, and hence $3^{m-2} > 3^{4\ell+1} > 1 + c + d$, so that equation (3.1) cannot hold in view of (3.10) and (3.11). Next, let $y = v_1 \otimes v_1 + v_2 \otimes v_2 \in V$. We have $(y, y) = 2 \neq 1 = (x, x)$. Thus $\langle y \rangle$ is a non-singular point in V which belongs to different S-orbits to $\langle x \rangle.S$. Let $V_1 = \langle v_1, v_2, \rangle$ and

$V_2 = V_1^{\perp}$. Then $sgnV_i = +$ and $V = V_1 \perp V_2$. Setting $I_i = I(V_i)$. We have $I_1 = \langle g_1, g_2 \rangle$, where

$$g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

It is easy to check that $yg_1 = \lambda^4 v_1 \otimes v_1 + \lambda^{-4} v_2 \otimes v_2 = -y$ and $yg_2 = v_2 \otimes v_2 + v_1 \otimes v_1 = y$. Hence $M_\Omega := (I_1 \times I_2) \cap \Omega(V) = S_{\langle y \rangle}$. By Proposition 4.1.6 in [29] that $M_\Omega \cong (\Omega_2^+(9) \times \Omega_{2\ell-2}^+(9)).2^2$. Therefore, $|\langle y \rangle S| = |S : M_\Omega| = \frac{1}{16} 9^{2\ell-2}(9^\ell - 1)(9^{\ell-1} + 1)$. If $\ell \neq 4$ or $\ell = 4$ but the triality τ of $\Omega_8^+(9)$ does not involved in G, then by Lemma 2.7.3 in [29], $Out(S) \cong D_8 \times \mathbb{Z}_2$ which is generated by $r_\square, r_\boxtimes, \delta_{\mathbf{f},\beta_0}(\lambda)$ and $\phi_\beta(\nu)$, where β_0 is a standard basis of V. As three of these generators fix $\langle y \rangle$, $|M_{\langle y \rangle}| \geqslant 2^3 |M_\Omega|$. Thus, in any cases, we have $|M : M_{\langle y \rangle}| \leqslant \frac{3.16}{8} |S : M_\Omega|$ Therefore $A \leqslant |\langle y \rangle M| \leqslant \frac{3}{8} 9^{2\ell-2}(9^\ell - 1)(9^{\ell-1} + 1)$. We have $A \leqslant \frac{3}{8} 9^{2\ell-2}(9^\ell - 1)(9^{\ell-1} + 1) \leqslant \frac{3}{4} 9^{2\ell-2}.9^{2\ell-1} \leqslant 9^{4\ell-3} = 3^{8\ell-6}$. Also $m - 2 - (8\ell - 6) = 2\ell^2 - 2 - 8\ell + 6 = 2\ell^2 - 8\ell + 4 = 2\ell(\ell - 4) + 4$. Thus if $\ell \geqslant 4$, then clearly $m - 2 > 8\ell - 6$, so that $3^{m-2} > 3^{8\ell-6} \geqslant A$, and equation (3.1) cannot hold by (3.10). Assume that $\ell = 3$. Then $A \leqslant \frac{1}{8} 9^4(9^3 - 1)(9^2 + 1)$. Now, by (3.10) again, it is enough to check that $\frac{1}{2}(3^{m-1} + 3) > \frac{1}{8} 9^4(9^3 - 1)(9^2 + 1) =: A_0$. As $m = 2.3^2 = 18$, $\frac{1}{2}(3^{m-1}+3) - A_0 = \frac{1}{8}(3^{18}+3^{17}+12) - \frac{1}{8}(3^{18}+3^{16}-3^{12}-3^8) = \frac{1}{8}(3^{17}-3^{14}+3^{12}+3^8+12) > 0$.

Case $\xi = -$. Let $x = u_1$ and $y = u_2$. Then $(x, x) = \mu^2 = -1$ and $(y, y) = 1$, so that x, y are non-singular vectors in V. As in previous case, we can check that the stabilizers of $\langle x \rangle$ and $\langle y \rangle$ are isomorphic to $\langle -1 \rangle \times \Omega_{2\ell-1}(9)$, so that $|\langle x \rangle S| = |\langle y \rangle S| = \frac{1}{2} 9^{\ell-1}(9^\ell - 1)$. The result follows as above. ■

Let K be a field and N be a finite dimensional vector space over M_Ω with a quadratic form Q and associated bilinear form $(.,.)$. Let $T^0(N) = M_\Omega, T^1(N) = N, T^2(N) = N \otimes N, \cdots, T^k(N) = \underbrace{N \otimes \cdots \otimes N}_{k}$, and define $T(N) = T^0(N) \oplus T^1(N) \oplus \cdots \oplus T^k(N) \oplus \cdots$. The bilinear map $T^r(N) \times T^s(N) \to T^{r+s}(N), (x, y) \mapsto x \otimes y, x \in T^r(N), y \in T^s(N)$ defines an operation on $T(N)$. Then $T(N)$ has a structure of an algebra and call *Tensor algebra*. Let

I be an ideal of T(N) generated by all elements $v \otimes v - Q(v).1$ for $v \in N$. The factor algebra $C(N) = C(N,Q) = T(N)/I$ is called the *Clifford algebra* of (N,Q). Identifying N with elements of degree 1 in $C(N)$, we have $v^2 = Q(v)$ and $uv + vu = (u,v)$ for any $u, v \in N$. Denote by $T_+(N)$ (resp. $T_-(N)$) the sum of all $T^k(N)$ for k even (resp. k odd). Then $T(N) = T_+(N) \oplus T_-(N)$. Also $I = I \cap T_+(N) \oplus I \cap T_-(N)$. Let $C_+(N) = T_+(N)/(I \cap T_+(N))$ and $C_-(N) = T_-(N)/(I \cap T_-(N))$. Then, $C(N) = C_+(N) \oplus C_-(N)$. The linear map $J : C(N) \to C(N)$ defined by $J(u) = u$ if $u \in C_+(N)$ and $J(u) = -u$ if $u \in C_-(N)$ is called the *main involution*. Also, there exists a unique anti-involution $\alpha : C(N) \to C(N)$ such that for any $v_1, v_2, \cdots, v_t \in N$, $(v_1 v_2 \ldots v_t)\alpha = v_t v_{t-1} \ldots v_1 \in C(N)$. We next describe a basis for $C(N)$ with respect to some fixed basis of N. Let $\beta = \{v_1, \cdots, v_n\}$ be a basis for N over M_Ω. Then for any increasing sequence $0 < i_1 < i_2 < \cdots < i_r \leqslant n$, the elements $v_{i_1} \cdots v_{i_r}$ together with 1 form a basis for $C(N)$, hence $dimC(N) = 2^n$. Also $C_+(N)$, (resp. $C_-(N)$) are subspace of $C(N)$ spanned by $v_{i_1} \cdots v_{i_r}$ with r even (resp. r odd).

We now assume that $dimN = n = 2\ell$ is even and (N, K, Q) is an orthogonal geometry of plus type. Let $\beta = \{e_1, \cdots e_\ell, f_1, \cdots, f_\ell\}$ be a standard basis for N. Denote by E and F the subspaces of N generated by $\{e_i\}_{i=1}^{\ell}$ and$\{f_i\}_{i=1}^{\ell}$, respectively. Then $N = E \oplus F$ and E, F are maximal totally singular subspace of N. Let $C(E)$ be the sub-algebra of $C(N)$ generated by E and set $C_+(E) = C(E) \cap C_+(N)$ and $C_-(E) = C(E) \cap C_-(N)$. As E is totally singular, $C(E)$ can be identified with the exterior algebra of E. Put $f = f_1 \cdots f_\ell$, then $C(N)f = C(E)f$. We can define a representation $\rho : C(N) \to End(C(E))$ by the condition $vuf = (\rho(v)u)f$ for $v \in C(N)$ and for all $u \in C(E)$.[6]

Proposition 3.64 (Proposition 5.4.9[29]) The spin representation embeds

(i)

$$B_\ell(q) \text{ in } \begin{cases} \Omega^+_{2^\ell}(q) & q \text{ even} \\ \Omega^+_{2^\ell}(q) & q \text{ odd}, \ \ell \equiv 0, 3 \pmod 4 \\ Sp_{2^\ell}(q) & q \text{ odd}, \ \ell \equiv 1, 2 \pmod 4). \end{cases}$$

(ii)

$$D_\ell(q) \text{ in } \begin{cases} \Omega^+_{2^{\ell-1}}(q) & \ell, q \text{ even} \\ \Omega^+_{2^{\ell-1}}(q) & q \text{ odd, } \ell \equiv 0 \pmod 4 \\ Sp_{2^{\ell-1}}(q) & q \text{ odd, } \ell \equiv 2 \pmod 4). \end{cases}$$

If ℓ is odd then the representation is not self-dual. The representation cannot be realized over a proper subfield of $\mathbf{F}_q$.

(iii)

$$^2D_\ell(q) \text{ in } \begin{cases} SU_{2^{\ell-1}}(q) & \ell \text{ odd} \\ \Omega^+_{2^{\ell-1}}(q^2) & \ell, q \text{ even} \\ \Omega^+_{2^{\ell-1}}(q^2) & q \text{ odd, } \ell \equiv 0 \pmod 4 \\ Sp_{2^{\ell-1}}(q^2) & q \text{ odd, } \ell \equiv 2 \pmod 4). \end{cases}$$

The representation cannot be realized over a proper subfield of $\mathbf{F}_{q^2}$.

Proposition 3.65 *Assume G is nearly simple primitive rank 3 of type $\Omega^\varepsilon_{2m}(3)$ and M is almost simple of type S, where $\widehat{S}$ is simply connected of type B_ℓ over $\mathbf{F}_{3^f}$. There is an M-orbit on $\mathfrak{E}^\varepsilon_\xi(V)$ such that equation (3.1) does not hold unless $(L, S, \lambda) = (O^-_{10}(3), O_5(3), 2\lambda_1)$, in which case M has 5 orbits on $\mathfrak{E}^-(V)$ and the equation holds for all M-orbits on $\mathfrak{E}^-_\xi(V)$ with $r = t$; or $(L, S, \lambda) = (O^+_8(3), O_7(3), \lambda_1)$ or $(O^+_{16}(3), O_9(3), \lambda_1)$, in which cases M has only one orbit on $\mathfrak{E}^+_\xi(V)$ so that $1^G_P \not\leqslant 1^G_M$ by Corollary 3.7, hence these cases appear in Table 1.2.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}^\varepsilon_\xi(V)$.

Case $f = 1$. We claim that if λ is a 3-restricted dominant weight with $dimL(\lambda)$ is even and greater than $dimN = 2\ell + 1$, then λ must be one of the following weights:

(i) $\lambda = \lambda_{\ell-1}, \ell$ even, $\ell \geqslant 4$, and $dimL(\lambda) = 2\ell^2 + \ell$;

(ii) $\lambda = 2\lambda_\ell, \ell = 6k, 6k+1$ or $6k+2$, for some non-negative integer k, and $dimL(\lambda) = 2\ell^2 + 3\ell, 2\ell^2 + 3\ell - 1, 2\ell^2 + 3\ell$, respectively;

(iii) $\lambda = \lambda_1, 2 \leqslant \ell \leqslant 8$, and $dimL(\lambda) = 2^\ell$ or $\lambda = 2\lambda_1$, $\ell = 2$ and $dimL(\lambda) = 10$.

Using (3.10) and (3.11), we have $3^{m-2} \leqslant |Aut(S)| = |Aut(\Omega_{2\ell+1}(3))| = 3^{\ell^2} \prod_{i=1}^{\ell}(3^{2i} - 1) \leqslant 3^{2\ell^2+\ell}$. Hence $dimL(\lambda) = 2m \leqslant 4\ell^2+2\ell+4$. Notice that if $\ell \geqslant 5$ then $\ell^3-4\ell^2-2\ell-4 \geqslant 5\ell^2-4\ell^2-2\ell-4 = (\ell-1)^2-5 \geqslant (5-1)^2-5 = 11 > 0$, and so $\ell^3 > 4\ell^2+2\ell+4$. If $\ell > 11$, then $dimL(\lambda) \leqslant 4\ell^2+2\ell+3 < \ell^3$, and hence by Theorem 5.1, in [38], λ is either $\lambda_{\ell-1}$ or $2\lambda_\ell$. For $2 \leqslant \ell \leqslant 11$, by Theorem 4.4 in [38] and the upper bound for dimension of $L(\lambda)$ above, again, λ is one of the weights above or case (iii) holds. It remains to get the restriction on ℓ in cases (i) and (ii). From the reference above, we also have $dimL(\lambda_{\ell-1}) = \ell(2\ell+1)$ and $dimL(2\lambda_\ell) = 2\ell^2+3\ell-\varepsilon_3(2\ell+1)$. Now case (i) holds as $dimL(\lambda_{\ell-1})$ is even if and only if ℓ is even. For case (ii), we can check that $\ell^2+3\ell-\varepsilon_3(2\ell+1)$ is even exactly when ℓ has the forms given in (ii).

We now consider case (i). As in Proposition 3.29, let $v = e_1\wedge x+\xi x\wedge f_1 = (e_1-\xi f_1)\wedge x$, where $\xi = \pm 1$. Since $Q(e_1 \wedge x) = 0 = Q(x \wedge f_1)$, we have $Q(v) = (e_1 \wedge x, x \wedge f_1) = \xi$. Hence v is non-singular. Let N_1 be the subspace of N generated by $\{e_1-\xi f_1, x\}$. As N_1 is non-degenerate, $N = N_1 \perp N_1^{\perp}$. Denote by H the centralizer of N_1 in $\Omega(N) \cong \Omega_{2\ell+1}(3)$. It follows that $H \cong \Omega_{2\ell-1}(3)$, and H fixes v. Using 3.10) and (3.11) we have $3^{m-2} \leqslant 1+c+d \leqslant |Aut(\Omega_{2\ell+1}(3)) : \Omega_{2\ell-1}(3)| = 2\cdot 3^{2\ell-1}(3^{2\ell}-1) \leqslant 3^{4\ell}$. Hence $m-2 < 4\ell$, so that $2\ell^2+\ell = 2m < 8\ell+4$. However as $\ell \geqslant 4$, $\ell^2+\ell-(8\ell+4) = 2\ell(2\ell-4)+(\ell-4) \geqslant 0$. Thus $m-2 \geqslant 4\ell+2$, a contradiction. For case (ii), let $v = e_1 \otimes e_i + \xi f_1 \otimes f_1$, where $\xi \in \{\pm 1\}$. Then $Q(v) = \xi$, hence v is non-singular in $L(2\lambda_\ell)$. Let $N_1 = \langle e_1, f_1\rangle$. Then N_1 is a non-degenerate subspace of N. As in case (i), let H be the centralizer of N_1 in $\Omega(N)$, as $H \cong \Omega_{2\ell-1}(3)$, we have $3^{m-2} \leqslant\leqslant |Aut(\Omega_{2\ell+1}(3)) : \Omega_{2\ell-1}(3)| < 3^{4\ell}$, hence $2m < 8\ell+4$. As $2m = dimL(2\lambda_\ell) = 2\ell^2+3\ell-\varepsilon_3(2\ell+1)$, and $\ell \geqslant 4$, $2m = 2\ell^2+3\ell-1 \geqslant 2\cdot 4\ell+3\ell-1 = 8\ell+(3\ell-1) \geqslant 8\ell+11 > 8\ell+4$. This gives a contradiction to $2m < 8\ell+4$.

For (iii), if the last case holds, that is $\lambda = 2\lambda_1$ and $\ell = 2$ then $S = \Omega_5(3) \leqslant \Omega_{10}^{\varepsilon}(3)$. We have $m = 5$ and by [13], we see that $\varepsilon = -$ and there are 5 orbits of non-singular points of each type with orbit sizes $4320, 2592, 2160, 540, 270$ and $(c, d) =$

$(2852, 1467), (1700, 891), (1412, 747), (332, 207), (152, 117)$, respectively. We see that $c - 2d = -3^{m-1} - 1$. Thus equation (3.8) holds so that equation (3.1) holds with $r = t$. For the spin representation $\lambda = \lambda_1$, by Proposition 3.64, $\widehat{S}$ fixes a non-degenerate symmetric form on $L(\lambda_1)$ if and only if $\ell \equiv 0, 3 \pmod 4$. Thus $\ell = 3, 4, 7, 8$. If $\ell = 3$ then we have an embedding $\Omega_7(3) \leqslant \Omega_8^+(3)$, and by [9] p.140, $O_7(3)$ has only one orbit on $\mathfrak{E}^+(V)$. If $\ell = 4$ then $\Omega_9(3) \leqslant \Omega_{16}^+(3)$. Moreover, by 4.6.3$(a)$ in [34], $\Omega_9(3)$ has only one orbit of each type of non-singular points on its spin module. If $\ell = 7$, then $B_7(3) \leqslant \Omega_{2^7}^+(3)$. However this is not a maximal embedding as $B_7(3) \leqslant D_8(3) \leqslant \Omega_{2^7}(3)$ by Proposition 3.64. In fact, the spin module for B_7 is the restriction of the spin module of D_8, in which B_7 is the stabilizer of a non-singular point in the natural module for D_8. Finally, $\ell = 8$. Then $B_8(3) \leqslant \Omega_{2^8}^+(3)$. Let V be the spin module and v_+, v_- be the maximal and minimal vectors correspondingly. Let P be the parabolic subgroup of $\widehat{S}$ which stabilizes v_+. Then P is of type $A_{\ell-1}$ and $P = U.L_1$, where U is the unipotent radical and L_1 is the Levi factor of P. Observe that L_1' centralizes v_+ and since $P^{op} = U^{op}L_1$ which fixes the point $\langle v_- \rangle$, it follows that L_1' fixes $z = v_+ + \xi v_-$ with $\xi = \pm 1$. Hence $L_1' \cong SL_\ell(3) \leqslant S_{\langle z \rangle}$. Thus $1 + c + d \leqslant |Aut(B_8(3)) : SL_\ell(3)| \leqslant 8.3^{36}(3^2+1)\cdots(3^8+1) \leqslant 16.3^{71} \leqslant 3^{74}$. Since $2m = 2^8$, $m - 2 = 2^7 - 2 = 126 > 74$, hence $3^{m-2} > 3^{74} \geqslant 1 + c + d$. Thus (3.1) cannot hold.

Case $f > 1$. Let Ψ be an even dimensional, self-dual irreducible kS-module. It follows from case $f = 1$ that $dim\Psi \geqslant 2^\ell$ if $2 \leqslant \ell \leqslant 6$ and $dim\Psi \geqslant 2\ell^2 + \ell$ otherwise. Note that $2^\ell \leqslant 2\ell^2 + \ell$ if and only if $\ell \leqslant 6$, and $2\ell^2 + 3\ell - \varepsilon_3(2\ell + 1) \geqslant 2\ell^2 + \ell$ for all $\ell \geqslant 2$. By Propositions 2.41 and 2.39, $dimV = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ. As $dimV = 2m$ is even, it follows that Ψ is an even dimensional, self-dual irreducible $k\widehat{S}$-module. As above, we have $3^{m-2} \leqslant |Aut(\Omega_{2\ell+1}(3^f))| < f \cdot 3^{f(2\ell^2+\ell)} \leqslant 3^{f(2\ell^2+\ell+1)}$. Hence $2m < 2f(2\ell^2 + \ell + 1) + 4$. If $\ell \geqslant 7$, then $2m \geqslant (2\ell^2 + \ell)^f$. Then by induction, we have $(2\ell^2 + \ell)^f > 2f(2\ell^2 + \ell + 1) + 4$, a contradiction. Thus $2 \leqslant \ell \leqslant 6$. Then $2m \geqslant 2^{f\ell}$. Hence

$2^{f\ell} < 2f(2\ell^2+\ell+1)+4$. This inequality holds except when $(\ell, f) = (2,2), (2,3), (3,2)$.

Case $\Omega_5(9)$. We have $S \cong Sp_4(9)$. If $dim\Psi > 4$, then $dim\Psi \geqslant 10$, and hence $dimV \geqslant 10^2 = 100$. If this is the case then clearly $3^{m-2} > |Aut(S)|$. Thus $dim\Psi = 4$ and V is the twisted tensor product $N \otimes N^{(1)}$. By [13], there exists an element of order 5 with non-singular eigenvectors of both types and has orbit size 99630 with $(c, d) = (65528, 34101)$ and we can check that equation (3.1) cannot hold.

Case $\Omega_7(9)$. If $dim\Psi > 8$, then $dim\Psi \geqslant 21$, and hence $dimV \geqslant 21^2 = 441$. Clearly $3^{m-2} > |Aut(S)|$ in this case. Thus $dim\Psi = 8$ and V is the twisted tensor product $N \otimes N^{(1)}$. However, since $\Omega_7(9) \leqslant \Omega_8^+(9) \leqslant \Omega_{64}^+(3)$, $N_G(S)$ is not maximal in G.

Case $\Omega_5(27)$. We have $S = P\Omega_5(3^3) \cong PSp_4(3^3)$. If $dim\Psi > 4$, then $dim\Psi \geqslant 10$, and hence $dimV \geqslant 10^2 = 100$. But we still have $3^{m-2} > |Aut(S)|$. Thus $dim\Psi = 4$ and $dimV = 4^3 = 64$. However, this case cannot occur as $\widehat{S}$ preserves no non-degenerate symmetric form on V. ■

Proposition 3.66 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^{\varepsilon}(3)$ and M is almost simple of type S, where $\widehat{S}$ is simply connected of type C_ℓ over $\mathbf{F}_{3^f}$, with $\ell \geqslant 3$. There is an M-orbit on $\mathfrak{E}_\xi^\varepsilon(V)$ such that equation (3.1) does not hold or $\ell = 5, 6, 7$, $f = 1$, and $\lambda = \lambda_{\ell-1}$ or $\ell = 3, f = 2, \lambda = (1+3^i)\lambda_\ell, 1 \leqslant i \leqslant 2$ and these cases appear in the Table 1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}_\xi^\varepsilon(V)$.

Case $f = 1$. Let $\lambda \in X_3$ be a 3-restricted dominant weight such that $L(\lambda) \cong V$. We first show that if $dimV > 2\ell$, then λ must be one of the following weights:

(i) $\lambda = \lambda_{\ell-1}$, $dimL(\lambda) = 2\ell^2 - \ell - 1 - \varepsilon_3(\ell)$;

(ii) $\lambda = 2\lambda_\ell$, ℓ even and $dimL(\lambda) = 2\ell^2 + \ell$;

(iii) $(\lambda, \ell, dimL(\lambda)) = (\lambda_1, 3, 14), (\lambda_2, 4, 40), (\lambda_3, 5, 110)$.

As $|Aut(PSp_{2\ell}(3))| = f \cdot 3^{\ell^2} \prod_{i=1}^{\ell}(3^{2i}-1) \leqslant 3^{2\ell^2+\ell}$, by (3.10) and (3.11), $3^{m-2} \leqslant 3^{2\ell^2+\ell}$, and hence $2m \leqslant 4\ell^2 + 2\ell + 4$. If $\ell \geqslant 5$ then $4\ell^2 + 2\ell + 4 < \ell^3$, and so $dimV < \ell^3$. Now, if $\ell > 11$, then $dimL(\lambda) = 2m < \ell^3$, and hence by Theorem 5.1, in [38], λ is either $\lambda_{\ell-1}$ or $2\lambda_\ell$. For $2 \leqslant \ell \leqslant 11$, by Theorem 4.4 in [38] and the upper bound for dimension of $L(\lambda)$ above, again, λ is one of the weights in (i), (ii) or $(\lambda, \ell, dimL(\lambda))$ are as in (iii). This proves our claims.

We have $dimL(2\lambda_\ell) = \ell(2\ell+1)$, $L(2\lambda_\ell) \cong S^2(V)$, and $dimL(\lambda_{\ell-1}) = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell)$, $L(\lambda_{\ell-1}) \cong w^\perp/(\langle w\rangle \cap w^\perp)$, where $w = e_1 \wedge f_1 + \cdots + e_\ell \wedge f_\ell$. In these cases, $\widehat{S}$ leaves invariant a quadratic form Q induced from the symplectic form on N. First, suppose that $\lambda = 2\lambda_\ell$. Let $v = e_1 \otimes e_1 + \xi f_1 \otimes f_1 \in V$. Since $dimL(2\lambda_\ell) = \ell(2\ell+1)$ is even, ℓ must be even. Let H be the centralizer in S of the subspace generated by $\{e_1, f_1\}$. Then $H \cong Sp_{2\ell-2}(3)$. By (3.10) and (3.11), we have $3^{m-2} \leqslant |Aut(PSp_{2\ell}(3)) : PSp_{2\ell-2}(3)| = \dfrac{2.3^{\ell^2}(3^{2\ell}-1)}{3^{(\ell-1)^2}} < 3^{4\ell}$. Hence $2m < 8\ell+4$. As $2m = \ell(2\ell+1)$, it follows that $\ell(2\ell+1) < 8\ell+4$. As $\ell \geqslant 3$ and ℓ is even, we conclude that $\ell \geqslant 4$. Then $\ell(2\ell+1) \geqslant 4(2\ell+1) = 8\ell+4$, a contradiction. Next, consider case $\lambda = \lambda_{\ell-1}$. As $dimL(\lambda_{\ell-1}) = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell)$ is even, $\ell = 6k-1, 6k$ or $6k+1$. Now $L(\lambda_{\ell-1})$ has a basis consisting of $e_i \wedge e_j, f_i \wedge f_j, 1 \leqslant i < j \leqslant \ell, e_i \wedge f_j, 1 \leqslant i \neq j \leqslant \ell$, and $e_i \wedge f_i - e_{i+1} \wedge f_{i+1}, i = 1, \cdots, \ell - 1 - \varepsilon_p(\ell)$. Let $z_\xi = e_1 \wedge e_2 + \xi f_1 \wedge f_2$, where $\xi = \pm$. Then z_ξ is non-singular and $Q(z_\xi) = -\xi$. Let $N_1 = \langle e_1, e_2, f_1, f_2\rangle$ be a subspace of N. Then $\wedge^2 N_1$ has a basis $\{e_1 \wedge e_2, f_1 \wedge f_2, e_1 \wedge f_2, e_2 \wedge f_1, e_1 \wedge f_1 - e_2 \wedge f_2\}$. Observe that z_ξ is of type ξ. Since N_1 is non-degenerate, $N = N_1 \perp N_1^\perp$. Let H, K be the centralizer in S of $N_1, N_1^\perp$, respectively. Then $H \cong Sp_{2(\ell-2)}(3)$, $K \cong Sp_4(3) \cong Spin_5(3)$, $K_{\langle z_\xi\rangle} \cong Spin_4^\xi(3)$ and H centralizes $\langle z_\xi\rangle$, so that $E := Spin_4^\xi(3) \times Sp_{2\ell-4}(3) \leqslant S_{\langle z_\xi\rangle}$, hence $1 + c + d \leqslant |Aut(S) : \overline{E}| = |Aut(PSp_{2\ell}(3)) : (\Omega_4^\xi(3) \times PSp_{2\ell-4}(3))| < 3^{8\ell-7}$. Hence $2m < 16\ell - 10$. Since $2m = 2\ell^2 - \ell - 1 - \varepsilon_p(\ell) \geqslant 2\ell^2 - \ell - 2$, we have $2\ell^2 - \ell - 2 < 16\ell - 10$, or equivalent $2\ell^2 - 17\ell + 8 < 0$. If $\ell \geqslant 8$, then $2\ell^2 - 17\ell + 8 = 2\ell^2 - 16\ell - (\ell - 8) = (2\ell-1)(\ell-8) \geqslant 0$, a contradiction. Thus $\ell < 8$. Since $\ell = 6k-1, 6k, 6k+1$, it follows that $\ell = 5, 6, 7$. If (ℓ, λ) are as in (iii), then by Appendix $A.2$ in [38], $ind(L(\lambda)) = -$, so that $\widehat{S}$ does not fix any non-degenerate quadratic form on $L(\lambda)$.

Case $f > 1$. Let Ψ be an even dimensional, self-dual irreducible $k\widehat{S}$-module. It follows from case $f = 1$ that either $dim\Psi = 2\ell$ or $dim\Psi \geqslant 2\ell^2 - \ell - 1 - \varepsilon_3(\ell) \geqslant 2\ell^2 - \ell - 2$. By Propositions 2.41 and 2.39, $dimV = (dim\Psi)^f$, for some such self-dual irreducible $k\widehat{S}$-module Ψ. By (3.10) and (3.11), we have $3^{m-2} \leqslant |Aut(PSp_{2\ell}(3^f))| < 3^{f(2\ell^2+\ell+1)}$. Hence $2m < 2f(2\ell^2 + \ell + 1) + 4$. We assume first that $dim\Psi = 2\ell$. Then $2m = (2\ell)^f$ and so $(2\ell)^f < 2f(2\ell^2+\ell+1)+4$. By induction, we can see that this holds only when $f = 2$. Now when $f = 2$, by Proposition 3.62, equation (3.1) cannot hold either. Finally, assume that $dim\Psi \geqslant 2\ell^2 - \ell - 2$. Then we have $(2\ell^2 - \ell - 2)^f < 2f(2\ell^2 + \ell + 1) + 4$. Using induction, this inequality cannot occur. Thus equation (3.1) cannot hold. ■

Proposition 3.67 *Assume G is nearly simple primitive rank 3 of type $\Omega^{\varepsilon}_{2m}(3)$ and M is almost simple of type S, where $\widehat{S}$ is simply connected of type D_ℓ or ${}^2D_\ell$ over $\mathbf{F}_{3^f}$. There is an M-orbit on $\mathfrak{C}^{\varepsilon}_{\xi}(V)$ such that equation (3.1) does not hold so that M is not in Tables 1.1-1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s,t\}$ and for any M orbits in $\mathfrak{C}^{\varepsilon}_{\xi}(V)$.

Case $f = 1$. Let $(N, \mathbf{F}, Q)$ be a classical orthogonal geometry of type ε with $\widehat{S} = \Omega(N)$. Let $\beta = \{e_1, \cdots, f_\ell\}$ be a standard basis for N. Let $\lambda \in X_3$ be a 3-restricted dominant weight such that $L(\lambda) \cong V$. We will show that if $dimV > 2\ell$, then λ must be one of the following weights:

(i) $\lambda = \lambda_{\ell-1}$, $dimL(\lambda) = 2\ell^2 - \ell, l$ even;

(ii) $\lambda = 2\lambda_\ell$, ℓ even and $dimL(\lambda) = 2\ell^2 + \ell - 1 - \varepsilon_3(\ell)$;

(iii) $\lambda = \lambda_1, \lambda_2, 4 \leqslant \ell \leqslant 10$;

(iv) $\ell = 4, \lambda = \lambda_1 + \lambda_2, \lambda_1 + \lambda_4, \lambda_2 + \lambda_4, dimL(\lambda) = 56$.

As $|Aut(P\Omega^{\varepsilon}_{2\ell}(3))| \leqslant 3^{2\ell^2-\ell+2}$, by (3.10) and (3.11), $3^{m-2} \leqslant 3^{2\ell^2-\ell+2}$, and hence $2m \leqslant 4\ell^2 - 2\ell + 8$. Since $\ell \geqslant 4$, $\ell^3 - (4\ell^2 - 2\ell + 8) = (\ell^2 + 2)(\ell - 4) \geqslant 0$, and so $dimV = 2m \leqslant \ell^3$. By Theorem 5.1, in [38], if $\ell > 11$, then λ is either $\lambda_{\ell-1}$ or $2\lambda_\ell$. For $4 \leqslant \ell \leqslant 11$, by

Appendix $A.41$ in [38] and the upper bound for dimension of $L(\lambda)$, λ is one of the weights above or λ appears in (iii) and (iv).

Firstly assume $\lambda = \lambda_{\ell-1}$. Then $V \cong L(\lambda) \cong \wedge^2 N$ and $dimV = 2\ell^2 - \ell = \ell(2\ell - 1)$. As $dimV = 2m$ is even, ℓ must be even. For $\xi = \pm$, let $z_\xi = (e_1 - \xi f_1) \wedge (e_2 + f_2) \in V$. Then z_ξ is non-singular in V. Let $U = \langle e_1 - \xi f_1, e_2 + f_2 \rangle$. We can check that $D(U) = -\xi$ and $sgn(U) = \xi$. It follows that $sgn(U^\perp) = \xi\varepsilon$. Let $H = \Omega(U^\perp)$. Then $H \cong \Omega^{\varepsilon\xi}_{2\ell-2}(3)$ and H fixes z_ξ, so that $H \leqslant \widehat{S}_{\langle z_\xi \rangle}$. Thus $|\langle z_\xi \rangle M| \leqslant |Aut(S) : H| \leqslant 3^{4\ell}$. Since $2m = 2\ell^2 - \ell$, $m - 2 = \frac{1}{2}(2\ell^2 - \ell - 4)$. By (3.10) and (3.11) again, $3^{m-2} \leqslant 3^{4\ell}$. This is equivalent to $m - 2 \leqslant 4\ell$, and so $\frac{1}{2}(2\ell^2 - \ell - 4) \leqslant 4\ell$ or equivalently $(\ell - 4)(2\ell - 1) \leqslant 8$. This inequality is satisfied only when $\ell = 4$. Suppose that $\ell = 4$. Then $m = 14$. Assume that $\varepsilon = +$. As $\Omega(U)$, $\Omega(U^\perp)$ and $r_{e_2+f_2}, r_{e_2-f_2}$ fix z_ξ, we have $|M : M_{\langle z_\xi \rangle}| \leqslant \frac{1}{4}|Out(S)|.|S : H| \leqslant 6|S : H|$, where $H = P\Omega(U) \otimes P\Omega(U^\perp) \cong P\Omega^\xi_2(3) \otimes P\Omega^\xi_6(3) \leqslant S_{\langle z_\xi \rangle}$ and $|Out(S)| \leqslant 24$ as $f = 1$. Thus $|M : M_{\langle z_\xi \rangle}| \leqslant \frac{1}{2}3^7(3^2+1)(3+\xi)(3^3+\xi) < 3^{13} = 3^{m-1}$. Assume that $\varepsilon = -$. As $\frac{1}{2}(q-1)\ell = 4$ is even, it follows from Proposition 2.6 that $D(Q) = \boxtimes$, and so by Proposition 2.8.2 in [29], $|Out(S)| = 4$. Therefore $|M : M_{\langle z_\xi \rangle}| \leqslant 4.|S : H|$ where $H \cong (\Omega^\xi_2(3) \otimes \Omega^{-\xi}_6(3)).2^2 \leqslant S_{\langle z_\xi \rangle}$ which is the stabilizer in S of U. Therefore $|M : M_{\langle z_\xi \rangle}| \leqslant \frac{1}{16}3^6(3^4+1)(3-\xi)(3^3+\xi) \leqslant \frac{1}{8}3^{14}$. However $\frac{1}{2}(3^{m-1} + 3) - \frac{1}{8}3^{14} = \frac{1}{2}(3^{13} + 3) - \frac{1}{8}3^{14} = \frac{1}{8}(4.3^{13} + 12 - 3^{14}) = \frac{1}{8}(3^{13} + 12) > 0$ and hence $|M : M_{\langle z_\xi \rangle}| < \frac{1}{2}(3^{m-1} + 3)$ which contradicts to (3.11).

Secondly assume that $\lambda = 2\lambda_\ell$. Then $dimL(\lambda) = 2\ell^2 + \ell - 1 - \varepsilon_3(\ell)$. Since $dimL(\lambda)$ is even, this implies that ℓ must have one of the following forms $6k - 1, 6k, 6k + 1$, where $k \geqslant 1$. It follows that $\ell \geqslant 5$, and so $|Out(S)| = 2df \leqslant 8$. As in Proposition 3.31, take $z_\xi = e_1 \otimes e_1 + \xi f_1 \otimes f_1 + \langle w \rangle \cap w^\perp$. Let H be the centralizer in S of $\langle e_1, f_1 \rangle$. Then H fixes z_ξ and $H \cong P\Omega^\varepsilon_{2\ell-2}(3)$. Thus $|M : M_{\langle z_\xi \rangle}| \leqslant |Aut(S) : H| \leqslant 3^{4\ell-1}$. Since $2m = 2\ell^2 + \ell - 1 - \varepsilon_3(\ell) \geqslant 2\ell^2 + \ell - 2$, we have $m - 2 \geqslant \frac{1}{2}(2\ell^2 + \ell - 6)$. Thus $\frac{1}{2}(2\ell^2 + \ell - 6) \leqslant 4\ell - 1$. But this is equivalent to $(\ell - 4)(2\ell + 1) \leqslant 0$, which is absurd as $\ell \geqslant 5$.

Next assume that $\lambda = \lambda_1$, and $4 \leqslant \ell \leqslant 10$ so that $V \cong L(\lambda_1)$ is the spin module. From

Proposition 3.64, we have $\varepsilon = +$ and $\ell \equiv 0 (\mathrm{mod}\ 4)$. Thus $\ell = 4, 8$. However, if $\ell = 4$, then $dimL(\lambda_1) = 2^3 = 8$, and so this case cannot happen. Thus $\ell = 8$ and $D_8(3) \leqslant \Omega^+_{2^7}(3)$. Argue as in Proposition 3.65, with $z_\xi = v_+ + \xi v_-$, we have $SL_8(3)$ fixes $\langle z_\xi \rangle$ so that $|M : M_{\langle z_\xi \rangle}| \leqslant 3^{59} < 3^{63} = 3^{m-1}$ as $m = 2^6 = 64$. This contradicts to (3.10).

Finally assume that case (iv) holds. Without lost, we can take $\lambda = \lambda_1 + \lambda_4$. Then $V = L(\lambda) \cong \wedge^3 N$, where N is the natural module for $\widehat{S}$, and $2m = 56$ so that $m = 28$. Let $z_\xi = e_1 \wedge e_2 \wedge e_3 + \xi f_1 \wedge f_2 \wedge f_3 \in V$. Then z_ξ is non-singular. Let K be the stabilizer of a totally singular decomposition $\langle e_1, e_2, e_3 \rangle \oplus \langle f_1, f_2, f_3 \rangle$ in $P\Omega(\langle e_1, \cdot, f_3 \rangle)$. Then $K \cong PGL_3(3).2$ and K fixes $\langle z_\xi \rangle$. Hence $|M : M_{\langle z_\xi \rangle}| \leqslant |Aut(P\Omega^\varepsilon_8(3)) : K| \leqslant |Out(S)||S : K| \leqslant 3^{23} < 3^{26} = 3^{m-2}$. Thus (3.1) cannot hold.

Case $f \geqslant 2$. Let Ψ be an even dimensional, self-dual irreducible $k\widehat{S}$-module. It follows from case $f = 1$ that $dim\Psi = 2\ell$ or $dim\Psi \geqslant 2\ell^2 - \ell$ when $\ell \geqslant 8$ or $dim\Psi = 2^{\ell-1}$ when $4 \leqslant \ell \leqslant 7$. By Propositions 2.41 and 2.39, $dimV = (dim\Psi)^f$, for some self-dual irreducible $k\widehat{S}$-module Ψ of even degree. If $dim\Psi = 2\ell$ then $2m = (2\ell)^f$. By (3.10) and (3.11), we have $3^{m-2} \leqslant |Aut(P\Omega^\varepsilon_{2\ell}(3^f))| < 3.3^{f(2\ell^2-\ell+1)}$ or equivalently $2^{f-1}\ell^f - 2 < 1 + f(2\ell^2 - \ell + 1)$. This happens only when $f = 2$. The result follows from Proposition 3.63. Assume that $dim\Psi > 2\ell$. If $\ell \geqslant 8$, then $dimV \geqslant (dim\Psi)^f \geqslant (2\ell^2 - \ell)^f$. As above, we have $\frac{1}{2}(2\ell^2 - \ell)^f - 2 < 1 + f(2\ell^2 - \ell + 1)$. Clearly this inequality cannot happen. Thus $4 \leqslant \ell \leqslant 7$. Then $dimV \geqslant (dim\Psi)^f \geqslant 2^{f(\ell-1)}$. Hence $2^{f(\ell-1)-1} - 2 < 1 + f(2\ell^2 - \ell + 1)$. This holds only when $\ell = 4$ and $f = 2$. In this case, $S = D_4(3^2)$ and Ψ is the spin representation of $\widehat{S}$. Since the spin representation of $\widehat{S}$ can be obtained from the natural representation of S via triality, (3.1) cannot hold by Proposition 3.63. The proof is now completed. ■

Proposition 3.68 *Assume G is nearly simple primitive rank 3 of type $\Omega^\varepsilon_{2m}(3)$ and M is almost simple of type S, where $\widehat{S}$ is of exceptional type above and define over $\mathbf{F}_{3^e}$ with $e \geqslant 1$. There is an M-orbit on $\mathfrak{E}^\varepsilon_\xi(V)$ such that equation (3.1) does not hold unless*

$(L, S, \lambda) = (P\Omega^{\varepsilon}_{52}(3), F_4(3), \lambda_1)$. *In this case M is in the Table 1.3.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}^{\varepsilon}_{\xi}(V)$.

(a) **Case** G_2. By (3.10) and (3.11), $3^{m-2} \leqslant |Aut(G_2(3^e))| \leqslant 3^{15e}$. Thus $dimV = 2m \leqslant 30e + 4$. Assume first that $e = 1$. Then $dimV = 2m \leqslant 34$. From Appendix $A.49$ in [38], $dimL(\lambda) > 500$, a contradiction. Assume that $e \geqslant 2$. By Propositions 2.41 and 2.39, $2m = (dim\Psi)^e$, for some self-dual irreducible $k\widehat{S}$-module Ψ of even degree. It follows that $2m \geqslant 500^e$, and hence $30e + 4 \geqslant 500^e$, a contradiction.

(b) **Case** F_4. By (3.10) and (3.11), $3^{m-2} \leqslant |Aut(F_4(3^e))| \leqslant 3^{53e}$. So $2m \leqslant 106e + 4$. Assume that $e = 1$. Then $dimV \leqslant 110$. It follows from Appendix $A.50$ in [38], $\lambda = \lambda_1$, $L(\lambda) = L(F_4)$, the simple Lie algebra of type F_4 over $\mathbf{F}_3$, and $dimL(\lambda) = 52$. In this case, we have an embedding $F_4(3) \leqslant \Omega^{\varepsilon}_{52}(3)$. Assume that $e \geqslant 2$. We have $2m \geqslant 52^e$, so that $52^e \leqslant 106e + 4$. But this cannot happen for any $e \geqslant 2$.

(c) **Case** ${}^{\varepsilon}E_6$. By (3.10) and (3.11), $3^{m-2} \leqslant |Aut({}^{\varepsilon}E_6(3^e))| \leqslant 3^{79e}$. Thus $2m \leqslant 158e+4$. Assume that $e = 1$. Then $dimV \leqslant 162$. From Appendix $A.51$ in [38], $dimL(\lambda) \geqslant 572$, a contradiction. Assume that $e \geqslant 2$. Then clearly $572^e > 158e + 4$.

(d) **Case** E_7. By (3.10) and (3.11), $3^{m-2} \leqslant |Aut(E_7(3^e))| \leqslant 3^{134e}$. Thus $2m \leqslant 268e+4$. If $e = 1$ then by Appendix $A.52$ and $A.2$ in [38], $dimL(\lambda) \geqslant 1330$, a contraction. Assume that $e \geqslant 2$. Clearly $1330^e > 268e + 3$.

(e) **Case** E_8. By (3.10) and (3.11), $3^{m-2} \leqslant |Aut(E_8(3^e))| \leqslant 3^{249e}$. Thus $2m \leqslant 498e+4$. Assume that $e = 1$. From Appendix $A.53$ in [38], $\lambda = \lambda_8$, $dimL(\lambda_8) = 248$ and $V = L(\lambda_8) = L(E_8)$. Let Φ denote the root system of E_8 and Π the fundamental system of roots of Φ. Let α_0 be the highest root of Φ. Then the element e_{α_0} in $L(E_8)$ is a maximal vector with high weight λ_8. Let $\xi = \pm 1$ and $z = e_{\alpha_0} + \xi e_{-\alpha_0} \in L(E_8)$. Then z is non-singular in V and clearly E_7 is contained in the stabilizer of $\langle z \rangle$. Hence $1+c+d \leqslant |E_8(3) : E_7(3)| < 3^{120}$. Since $dimV = 248$, we have $m = 124$, so that $3^{m-2} = 3^{122} > 3^{120}$. Therefore equation (3.1) cannot hold. Assume that $e \geqslant 2$. Clearly $248^e > 498e + 3$ for any $e \geqslant 2$.

(f) **Case** $^3D_4$. By (3.10) and (3.11), $3^{m-2} \leqslant |Aut(^3D_4(3^e))| \leqslant 3^{30e}$. Thus $2m \leqslant 60e+4$. If $e = 1$ then $dimV \leqslant 64$. From Appendix $A.53$ in [38], $\lambda = \lambda_4$ and $dimL(\lambda_4) = 8$, or $\lambda = \lambda_3, dimL(\lambda_3) = 28$ or $\lambda = \lambda_1 + \lambda_2, dimL(\lambda_1 + \lambda_2) = 56$. Observe that $\lambda_1, \lambda_1 + \lambda_2$ are not invariant under the triality of D_4, but λ_3 is invariant. Firstly, assume that $\lambda = \lambda_1$. Since the splitting field for $^3D_4(3)$ is $\mathbf{F}_{3^3}$, we have embedding $^3D_4(3) \leqslant \Omega_8^+(3^3) \leqslant \Omega_{24}^+(3)$ where the latter is the over-field subgroup embedding. Thus this cannot happen. For the remaining cases, the same argument will lead to a contradiction. Finally, $\lambda = \lambda_3$. Then $L(\lambda_3)$ can be realized over $\mathbf{F}_3$, hence we can take $V \cong L(\lambda_3)$. If N is the natural module for $O_8^+(3^3)$ then $V \cong \wedge^2 N$. However this still does not gives rise to maximal embedding as $^3D_4(3) \leqslant \Omega_8^+(3^3) \leqslant \Omega_{28}^+(3)$. If $e \geqslant 2$ then by Proposition 5.4.8 in [29], $dimV \geqslant 24^e$. Clearly $24^e > 60e + 4$ for any $e \geqslant 2$.

(g) **Case** $^2G_2$. By (3.10) and (3.11), $3^{m-2} \leqslant |Aut(^2G_2(3^{2e+1}))| \leqslant 3^{8(2e+1)}$. Thus $2m \leqslant 32e + 20$. By Proposition 2.41, $dimV \geqslant 7^{2e+1}$, and so $7^{2e+1} \leqslant 32e + 20$, where $e \geqslant 1$. We can check that this cannot hold. ∎

Embedding of Sporadic groups

Proposition 3.69 *Assume G is nearly simple primitive rank 3 of type $\Omega_{2m}^{\varepsilon}(3)$ and M is almost simple of type S, where S is a simple sporadic group. There is an M-orbit on $\mathfrak{E}_{\xi}^{\varepsilon}(V)$ such that equation (3.1) does not hold unless $(\widehat{S}, L) = (2.Co_1, \Omega_{24}^+(3))$, in which case M has only 2 orbits on $\mathfrak{E}_{\xi}^{+}(V)$ for only one type of points and so $1_P^G \not\leqslant 1_M^G$ by Corollary 3.7. In this case M is in Table 1.2.*

Proof. Assume equation (3.1) holds for some $r \in \{s, t\}$ and for any M orbits in $\mathfrak{E}_{\xi}^{\varepsilon}(V)$. By (3.10) and (3.11), we have $|Aut(S)| \geqslant 3^{m-2}$, hence $2m \leqslant 2log_3(|Aut(S)|) + 4 = g_3(S)$. Since V is an absolutely irreducible $\mathbf{F}_3\widehat{S}$-module with $ind(V) = +$ and $dimV$ is even, by Lemma 3.33 and [19], we only need to consider the following cases: $(S, dimV) = (M_{11}, 10)$, $(M_{12}, 10)$, $(M_{23}, 22)$, $(M_{24}, 22)$, $(Co_3, 22)$, $(Co_1, 24)$. Observe first that the first four cases do not give rise to maximal embedding since $M_n \leqslant A_n \leqslant P\Omega^{\varepsilon}_{n-1-\varepsilon_3(n)}(3)$, where the last embedding is obtained via the fully deleted permutation module for A_n. Therefore we need to consider the last two cases. Assume that $S = Co_1$ and $dimV = 24$. It follows that $m = 12$, and $sgnV = +$ by [13]. Observe that Co_2 is a maximal subgroup of Co_1 and it is also a stabilizer of a non-singular point, say x. Then $|xS| = 98280 < 3^{m-1} = 177147$ and (3.1) cannot hold in view of (3.10). For other type of points, inside Co_2, the maximal subgroup $HS : 2$ of Co_2 fixes a non-singular point y, by checking the orders, we can see that $S_y = HS : 2$ and so $|yS| = 46872483840$. Next choose an $2A$-element g in Co_1. Then the normalizer of g in Co_1 is $H := 2^{1+8}_{+} {}^{\cdot} O_8^+(2)$. Choose a non-singular point z of the same type as that of y in the eigenspace of g of dimension 8. We have $|H_z| = 165150720$. We now look for an involution h in Co_2 not contained in H which fixes z. Then $\langle H, h\rangle$ is exactly the stabilizer of z in Co_1, and $|zS| = 165150720$. In fact the stabilizer of z in Co_1 is isomorphic to $2^{11} : M_{23}$ which is a subgroup of index 24 in the maximal subgroup $2^{11} : M_{24}$ of Co_1. Clearly these are two distinct S-orbits of the same type. As $|yS| + |zS| = |\mathfrak{E}^{\varepsilon}(V)| = \frac{1}{2}3^{11}(3^{12} - 1)$, S has only two orbits on $\mathfrak{E}^{\varepsilon}(V)$, hence (3.1) holds by Corollary 3.7. Finally assume that $S = Co_3$ and $dimV = 22$. We have $m = 11$ and by [13], we have $sgnV = +$. We also see that $McL : 2$ is the stabilizer of a non-singular point x with orbit size $|xS| = 276 < 3^{m-1}$. For other type of non-singular points, we can find a non-singular point y with $|yS| = 23054625$, and we see that (3.1) cannot hold. ■

Appendix A
Classification of nearly simple primitive rank 3 groups

Let G be a primitive rank 3 group of finite degree n. Then one of the following holds:
(i) $T \times T \lhd G \leqslant T_0 \wr \mathbb{Z}_2$, where T_0 is a $2-$transitive group of degree n_0, the socle T of T_0 is simple and $n = n_0^2$;
(ii) G is an affine group, i.e. G has a regular elementary abelian normal subgroup and n is a prime power;
(iii) $L := soc(G)$ is simple.

Combining results of Kantor and Liebler ([28]), Liebeck and Saxl ([37]) and Bannai ([3]), we get the list of all nearly simple primitive rank three groups:

Theorem A.1 (1) *Let L be one of the groups*

$$Sp_{2m-2}(q), \Omega^{\pm}_{2m}(q), \Omega_{2m-1}(q) \text{ or } SU_m(q)$$

for $m \geqslant 3$ and q a prime power. Let $L \trianglelefteq G$ with $G/Z(L) \leqslant Aut(L/Z(L))$. Assume that G acts as a primitive rank 3 permutation group on the set $\mathfrak{E}$ of cosets of a subgroup P of G. Then at least one of the following holds up to conjugation under $Aut(L/Z(L))$.
(i) $\mathfrak{E}$ is an L-orbit of singular (or isotropic) points;
(ii) $\mathfrak{E}$ is an L-orbit of maximal totally singular (or isotropic) subspaces and L is one of the following groups: $Sp_4(q), SU_4(q), SU_5(q), \Omega^-_6(q), \Omega^+_8(q)$ or $\Omega^+_{10}(q)$;
(iii) $\mathfrak{E}$ is any L-orbit of non-singular points and $L = SU_m(2), \Omega^{\pm}_{2m}(2), \Omega^{\pm}_{2m}(3)$ or $\Omega_{2m-1}(3)$;
(iv) $\mathfrak{E}$ is either orbit of nonsingular hyper-planes of $L = \Omega_{2m-1}(4)$ or $\Omega_{2m-1}(8)$, where $G = \Omega_{2m-1}(8) \cdot 3$ in the latter case;
(v) $L = SU_3(3), L \cap P = PSL_3(2)$;
(vi) $L = SU_3(5), L \cap P = 3 \cdot A_7$;
(vii) $L = SU_4(3), L \cap P = 4 \cdot PSL_3(4)$;
$(viii)$ $L = Sp_6(2), P = G_2(2)$;
(ix) $L = \Omega_7(3), L \cap P = G_2(3)$;
(x) $L = SU_6(2), L \cap P = 3 \cdot PSU_4(3) \cdot 2$;

(2) *Let $L = PSL_n(q) \leqslant G \leqslant Aut(L)$. Assume that G acts as a primitive rank 3 permutation group on the set $\mathfrak{E}$ of cosets of a subgroup P of G. Then one of the following*

Table A.1: L is an exceptional group.

n	L	$L \cap P$	$k; l$	$\{f_s; f_t\}$	comment
351	$G_2(3)$	$U_3(3) \cdot 2$	$126, 224$	$168; 182$	two classes
416	$G_2(4)$	J_2	$100, 315$	$65; 350$	
2016	$G_2(4)$	$U_3(4) \cdot 2$	$975, 1040$	$650; 1365$	two classes
130816	$G_2(8)$	$SU_3(8) \cdot 2$	$32319, 98496$	$18468; 112347$	$G = G_2(8) \cdot 3$
n_0	$E_6(q)$	D_5-parabolic	k_0, l_0	f_0, g_0	two classes

Table A.2: $G \cong A_m$

m	$n = [A_m : P]$	$\|P\|$	Structure of P	Decomposition of $(1_P)^{A_m}$
$m \geqslant 5$	$\frac{1}{2}m(m-1)$	$(m-2)!$	$H(\Sigma_2) \cap A_m$	$[m] + [m-1, 1] + [m-2, 2]$
6	15	24	$H(\Pi_2) \cap A_6$	$[6] + [4, 2] + [3^2]$
8	35	576	$H(\Pi_4) \cap A_8$	$[8] + [6, 2] + [4^2]$
9	120	1512	$P\Gamma L_2(8)$	$[9] + [4^2, 1] + [5, 1^4]_1$
10	1260	1440	$H(\Pi_5) \cap A_{10}$	$[10] + [8, 2] + [6, 4]$

holds up to conjugation under $Aut(L)$.
(i) $\mathfrak{E}$ is the set of lines for $L, n \geqslant 4$;
(ii) $L = PSL_2(4) \cong PSL_2(5), |\mathfrak{E}| = \binom{5}{2}$,
$L = PSL_2(9) \cong A_6, |\mathfrak{E}| = \binom{6}{2}$,
$L = PSL_4(2) \cong A_8, |\mathfrak{E}| = \binom{8}{2}$, *or*
$G = P\Gamma L_2(8), |\mathfrak{E}| = \binom{9}{2}$;
(iii) $L = PSL_3(4), L \cap P = A_6$;
(iv) $L = PSL_4(3), L \cap P = PSp_4(3)$.

(3) *Let G be a nearly simple primitive rank 3 group of exceptional Lie type, that is, $L/Z(L) \trianglelefteq G/Z(L) \leqslant Aut(L/Z(L))$, L is an exceptional group of Lie type and G acts as a primitive rank 3 permutation group on the set $\mathfrak{E}$ of cosets of a subgroup P of G. Then $L, L \cap P$ are given in Table A.1.*

$n_0 = (q^{12} - 1)(q^9 - 1)/(q^4 - 1)(q - 1)$;
$k_0 = q(q^8 - 1)(q^3 + 1)/(q - 1), l_0 = q^8(q^5 - 1)(q^4 + 1)/(q - 1)$;
$f_0 = q(q^9 - 1)(q^4 + 1)/(q^3 - 1), g_0 = q^2(q^6 + 1)(q^5 - 1)(q^4 + 1)/(q - 1)$.

(4) $L = A_m, m \geqslant 5$. *Let G is a nearly simple primitive rank 3 group of Alternating type $A_m, m \geqslant 5$ with point stabilizer P. Then m and P are given in Table A.2 and A.3. We denote by $H(\Sigma_r)$ (resp. $H(\Sigma_r) \cap A_m$) the subgroup of S_m (resp. A_m) which consists of the elements fixing some given subset Σ_r with $r < [m/2]$. We also denote by $H(\Pi_u)$ (resp. $H(\Pi_u) \cap A_m$) the subgroup of S_m (resp. A_m) which consists of elements permuting a fixed nontrivial complete set of blocks π_i with $|\pi_i| = u$.*

(5) *The list of nearly simple primitive rank 3 groups of sporadic type is in Table A.4.*

Table A.3: $G \cong S_m$

m	$n = [S_m : P]$	$\lvert P\rvert$	Structure of P	Decomposition of $(1_P)^{S_m}$
$m \geqslant 5$	$\frac{1}{2}m(m-1)$	$2(m-2)!$	$H(\Sigma_2)$	$[m] + [m-1, 1] + [m-2, 2]$
6	15	48	$H(\Pi_2)$	$[6] + [4, 2] + [2^3]$
8	35	1152	$H(\Pi_4)$	$[8] + [6, 2] + [4^2]$
10	1260	2880	$H(\Pi_5)$	$[10] + [8, 2] + [6, 4]$

Table A.4: L is a sporadic simple group

L	$P \cap L$	n	$k; l$	$\{f_s; f_t\}$	comment
M_{11}	$M_9.2$	55	$18; 36$	$10; 44$	
M_{12}	$M_{10}.2$	66	$20; 45$	$11; 54$	two classes
M_{22}	$2^4 \cdot A_6$	77	$16; 60$	$21; 55$	
M_{22}	A_7	176	$70; 105$	$21; 154$	two classes
M_{23}	$M_{21} \cdot 2$	253	$42; 210$	$22; 230$	
M_{23}	$2^4 \cdot A_7$	253	$112; 140$	$22; 230$	
M_{24}	$M_{22} \cdot 2$	276	$44; 231$	$23; 252$	
M_{24}	$M_{12} \cdot 2$	1288	$495; 792$	$252; 1035$	
J_2	$PSU_3(3)$	100	$36; 63$	$36; 63$	
HS	M_{22}	100	$22; 77$	$22; 77$	
McL	$PSU_4(3)$	275	$112; 162$	$22; 252$	
Suz	$G_2(4)$	1782	$416; 1365$	$780; 1001$	
Co_2	$PSU_6(2) \cdot 2$	2300	$891; 1408$	$275; 2024$	
Ru	${}^2F_4(2)$	4060	$1755; 2304$	$783; 3276$	
Fi_{22}	$2 \cdot PSU_6(2)$	3510	$693; 2816$	$492; 3080$	
Fi_{22}	$\Omega_7(3)$	14080	$3159; 10920$	$429; 13650$	two classes
Fi_{23}	$2 \cdot Fi_{22}$	31671	$3510; 28160$	$782; 30888$	
Fi_{23}	$P\Omega_8^+(3) \cdot S_3$	137632	$28431; 109200$	$30888; 106743$	
Fi_{24}'	Fi_{23}	306936	$31671; 275264$	$57477; 249458$	

Appendix B
Nearly simple groups of type $\Omega^{\varepsilon}_{2m}(2)$, $SU_m(2)$ and sporadic

We also obtain similar results when G is nearly simple primitive rank 3 of type L, where L is either $\Omega^{\varepsilon}_{2m}(2), m \geqslant 3$, or $SU_m(2), m \geqslant 4$ and G acts on the L-orbit $\mathfrak{E}(V)$ of non-singular points in the natural module V for L. The proof of Theorem B.1 is omitted.

Theorem B.1 *Let L be one of the following groups $\Omega^{\varepsilon}_{2m}(2), m \geqslant 4$ or $SU_m(2), m \geqslant 4$, and G be nearly simple primitive rank 3 of type L. Let P be the stabilizer of a non-singular point in V. Let M be any maximal subgroup of G. Then $1_P^G \leqslant 1_M^G$ unless the pairs (L, M) appear in Tables B.1-B.3.*

For reference purposes, we include here the similar results for almost simple primitive rank 3 groups of type S, where S is a simple sporadic group. The proof relies on [13].

Theorem B.2 *Let L be a simple sporadic group and G be almost simple primitive rank 3 of type L. Let P be a maximal subgroup of G such that G acts as a primitive rank 3 permutation groups on the set of cosets of P. Let M be any maximal subgroup of G. Then $1_P^G \leqslant 1_M^G$ unless the triples (G, P, M) appear in Tables B.4-B.7.*

As all the permutation characters of maximal subgroups of groups in Theorem B.2 are stored in [13], except $Fi_{22}.2, HS.2$ and $Fi'_{24}.2$, we can verify the Tables in Theorem B.2 easily. For the remaining groups, we will use the package 'atlasrep' to get the maximal subgroups and then apply equation (3.1) to determine the character containment.

The following GAP code will produce a list of maximal subgroups together with their permutation characters of a group with identifier "name".

```
gap>permchars:=function(name)
>local output, id, tbl, maxtab, chi, max, i;
> tbl :=CharacterTable(name);output := [ ];
> if HasMaxes(tbl)=false then Print('fail'); return fail; else max :=Maxes(tbl); fi;
> for i in [1..Size(max)] do maxtab :=CharacterTable(max[i]);
> id :=TrivialCharacter(maxtab); chi :=InducedClassFunction(maxtab, id, tbl);
> output[i] := [max[i],PermCharInfo(tbl, chi).ATLAS [1]]; od;
> return output; > end;
```

Table B.1: $M \in \mathcal{C}$

L	type of M	conditions	Remarks	orbits
$\Omega^{\varepsilon}_{2m}(2)$	P_α	$1 \leqslant \alpha \leqslant m$		$\leqslant 2$
	$O^-_{2a} \wr S_2$	$\varepsilon = +$	$m = 2a$	2
	$O^-_2 \wr S_4$	$\varepsilon = +$	$m = 4$	2
	$O^+_{2b}(2^\alpha)$	$b \geqslant 2, \alpha = 2, 3$	$m = b\alpha$	$\alpha - 1$
	$O^-_{2b}(2^3)$	$b \geqslant 2$	$m = 3b$	2
	$O^-_{2b}(2^2)$	$b \geqslant 2$	$m = 2b$	$\leqslant 2$
	$GU_m(2)$			1
$SU_n(2)$	$GU_1(2) \times GU_{n-1}(2)$			
	P_α	$1 \leqslant \alpha \leqslant n/2$		2
	$GU_1(2) \wr S_4$		$n = 4$	2
	$GU_a(2) \wr S_2$		$n = 2a$	2
	$GU_a(2^3)$		$n = 3a$	2
	$Sp_n(2)$		n even	1

Table B.2: $M \in \mathcal{S}$

L	socle of M	modules	Remarks	orbits
$\Omega^+_{n-1-\varepsilon_2(n)}(2)$	$A_n, n = 7, 9, 16$	$\lambda = (n-1, 1)$		2
$\Omega^-_{10}(2)$	A_{12}	$\lambda = (n-1, 1)$		2
$\Omega^-_8(2)$	$L_2(7) \cong L_3(2)$		$r = s$	3
$\Omega^-_{14}(2)$	$G_2(3)$			2
$\Omega^-_{14}(2)$	$U_3(3)$		$r = s$	6
$\Omega^+_{20}(2)$	$L^{\varepsilon}_6(2)$	λ_3		2
$\Omega^-_{14}(2)$	$PSp_6(2)$	λ_2	$r = s$	3
$\Omega^+_8(2)$	$PSp_6(2)$	Spin module		1
$\Omega^+_{16}(2)$	$PSp_8(2)$	Spin module		1
$\Omega^+_{22}(2)$	Co_2	Leech lattice		2
$\Omega^+_{24}(2)$	Co_1	Leech lattice		1
$SU_6(2)$	M_{22}			2
$SU_9(2)$	J_3			2

Table B.3: Exceptions

L	socle of M	modules	Remarks
$\Omega^{\varepsilon}_{2m}(2)$	$PSp_{2\ell}(2)$	$\lambda_{\ell-1}$	$\ell = 5,6,7$
$\Omega^{+}_{2\ell}(2)$	$PSp_{2\ell}(2)$	Spin module	$\ell = 5,6$
$\Omega^{+}_{32}(2)$	$P\Omega^{+}_{12}(2)$	Spin module	
$\Omega^{-}_{26}(2)$	$F_4(2)$	adjoint module	
$\Omega^{-}_{78}(2)$	$E_6(2)$	adjoint module	
$\Omega^{+}_{56}(2)$	$E_7(2)$	minimal module	
$\Omega^{-}_{28}(2)$	Ru		
$SU_9(2)$	$3^4.Sp_4(3)$		$M \in \mathcal{C}_6(G)$
$SU_{40}(2)$	$S_8(3)$	Weil module	
$SU_n(2)$	$Chev(2^f)$		

Table B.4: Sporadic Groups

G	P	M	1_P^G	Remarks	orbits
M_{11}	$3^2:Q_8.2$		$1a+10a+44a$		
		$A_6{\cdot}2$			2
		$L_2(11)$			1
M_{12}	$M_{10}:2$		$1a+11a+54a$		
		M_{11}		$1a+11a$	2
		M_{11}		$1a+11b$	1
		$M_{10}:2$		$1a+11b+54a$	2
		$L_2(11)$			1
		$M_9:S_3$			2
		$2\times S_5$			3
		$4^2:D_{12}$			3
		$A_4\times S_3$			3
M_{12}	$M_{10}:2$		$1a+11b+54a$		
		M_{11}		$1a+11a$	1
		M_{11}		$1a+11b$	2
		$M_{10}:2$		$1a+11a+54a$	2
		$L_2(11)$			1
		$M_9:S_3$			2
		$2\times S_5$			3
		$4^2:D_{12}$			3
		$A_4\times S_3$			3

Table B.5: Sporadic Groups (continue)

G	P	M	1_P^G	Remarks	orbits
M_{22}	$2^4 : A_6$		$1a + 21a + 55a$		
		$L_3(4)$		$1a + 21a$	2
		A_7		$1a + 21a + 154a$	2
M_{22}	A_7		$1a + 21a + 154a$		
		$L_3(4)$		$1a + 21a$	2
		$2^4 : A_6$		$1a + 21a + 55a$	2
$M_{22}.2$	$2^4 : S_6$		$1a + 21a + 55a$		
		$L_3(4).2_2$		$1a + 21a$	2
		$L_2(11).2$			2
M_{23}	$L_3(4) : 2_2$		$1a + 22a + 230a$		
		M_{22}		$1a + 22a$	2
		$23 : 11$			1
M_{23}	$2^4 : A_7$		$1a + 22a + 230a$		
		M_{22}		$1a + 22a$	2
		$23 : 11$			1
M_{24}	$M_{22} : 2$		$1a + 23a + 252a$		
		M_{23}		$1a + 23a$	2
		$M_{12} : 2$		$1a + 252a + 1035a$	2
		$2^6 : 3{\cdot}S_6$			2
		$2^6 : (L_3(2) \times S_3)$			2
		$L_2(23)$			1
		$L_2(7)$			4
M_{24}	$M_{12} : 2$		$1a + 252a + 1035a$		
		M_{23}		$1a + 23a$	1
		$M_{22} : 2$		$1a + 23a + 252a$	2
		$2^4 : A_8$			2
		$L_3(4) : S_3$			2
		$L_2(23)$			4
HS	M_{22}		$1a + 22a + 77a$		
		$U_3(5) : 2$		$1a + 175a$, 2 classes	1
		S_8			2
		$4{\cdot}2^4 : S_5$			2
		$2 \times A_6{\cdot}2^2$			2
		$5 : 4 \times A_5$			1
$HS.2$	$M_{22} : 2$		$1a + 22a + 77a$		
		$S_8 : 2$			2
		$2_+^{1+6} : S_5$			2
		$(2 \times A_6 \cdot 2^2).2$			2
		$5_+^{1+2} : [2^5]$			1
		$5 : 4 \times S_5$			1

Table B.6: Sporadic Groups (continue)

G	P	M	1_P^G	Remarks	orbits
J_2	$U_3(3)$		$1a+36a+63a$		
		$3{\cdot}PGL_2(9)$			2
		$2_-^{1+4}:A_5$			2
		$A_4\times A_5$			2
		$A_5\times D_{10}$			1
		$5^2:D_{12}$			2
$J_2.2$	$U_3(3):2$		$1a+36a+63a$		
		$3{\cdot}A_6\cdot 2^2$			2
		$2_-^{1+4}:S_5$			2
		$(A_4\times A_5):2$			2
		$(A_5\times D_{10}):2$			1
		$5^2:(4\times S_3)$			1
M^cL	$U_4(3)$		$1a+22a+252a$		
		$3_+^{1+4}:2S_5$			2
		$2\cdot A_8$			2
		$5_+^{1+2}:3:8$			2
$M^cL.2$	$U_4(3).2_3$		$1a+22a+252a$		
		$U_3(5):2$			2
		$3_+^{1+4}:4S_5$			2
		$2{\cdot}S_8$			2
		$5_+^{1+2}:(24:2)$			2
Suz	$G_2(4)$		$1a+780a+1001a$		
		$3_2\cdot U_4(3):2_3$			2
		$U_5(2)$			1
		$2_-^{1+6}\cdot U_4(2)$			2
		$3^5:M_{11}$			1
		$M_{12}:2$			2
		$3^{2+4}:2(A_4\times 2^2).2$			2
$Suz:2$	$G_2(4):2$		$1a+780a+1001a$		
		$3_2\cdot U4(3).(2^2)_{133}$			2
		$U_5(2):2$			1
		$2_-^{1+6}\cdot U_4(2).2$			2
		$3^5:(M_{11}\times 2)$			1
		$M_{12}:2\times 2$			2
		$3^{2+4}:2(S_4\times D_8)$			2

Table B.7: Sporadic Groups (continue)

G	P	M	1_P^G	Remarks	orbits
Co_2	$U_6(2):2$		$1a+275a+2024a$		
		McL			2
Ru	${}^2F_4(2)$		$1a+783a+3276a$		
		$(2^6:(U_3(3)):2$			3
		$(2^2\times Sz(8)):3$			2
		$2^{3+8}:L_3(2)$			3
		$U_3(5):2$			3
		A_8			4
		$L_2(29)$			1
Fi_{22}	$2\cdot U_6(2)$		$1a+429a+3080a$		
		$O_7(3)$		2 classes	2
		$O_8^+(2):S_3$			2
		${}^2F_4(2)'$			1
$Fi_{22}:2$	$2\cdot U_6(2)$		$1a+429a+3080a$		
		$G_2(3):2$			2
		$O_8^+(2):S_3\times 2$			2
		${}^2F_4(2)$			1
Fi_{22}	$O_7(3)$		$1a+429a+13650a$		
		$2\cdot U_6(2)$			2
		$O_8^+(2):S_3\times 2$			2
		$2^{10}M_{22}$			2
		${}^2F_4(2)'$			2
Fi_{23}	$2\cdot Fi_{22}$		$1a+782a+30888a$		
		$O_8^+(3):S_3$			2
		$3_+^{1+8}.2_-^{1+6}.3_+^{1+2.2S_4}$			2
		$3^3.[3^7].(2\times L_3(3))$			2
Fi_{23}	$O_8^+(3):S_3$		$1a+30888a+106743a$		
		$2\cdot Fi_{22}$			2
		$2^2\cdot U_6(2).2$			3
		$2^{11}\cdot M_{23}$			2

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